Acta Technica Napocensis: Civil Engineering & Architecture Vol. 56, No. 1, (2013)
Journal homepage: http://constructii.utcluj.ro/ActaCivilEng

Dynamic Response of Beams on Elastic Foundation with Axial Load

Iacob Borş[1], Tudor Milchiş[*2],

[123] *Technical University of Cluj-Napoca, Faculty of Civil Engineering. 15 C Daicoviciu Str., 400020, Cluj-Napoca, Romania*

Received 15 February 2013; Accepted 11 July 2013

Abstract

The vibrations of Euler-Bernoulli beams on elastic foundations, in Winkler`s model, submitted to axial load and dynamic external forces are study. The elastic beam is considered to have continuous mass witch concludes that systems have ∞ dynamical degree of freedom. The presents of axial load determines geometrically nonlinear vibrations. The six fundamental cases are highlighted, with the solutions of forth order linear and homogenous differential equations of vibrations represents the mode shapes functions and natural frequencies. To solve the differential equation, determined with the separations of variables method concludes in a Sturm-Lioville problem. A decrease of natural frequencies values can be observed, which corresponds to a decrease rigidity of compressed beams and the presents of elastic soil in compare to uncompressed ones.

Rezumat

Vibraţiile libere a barelor pe mediu elastic sunt studiate în prezenta lucrare. Barele se consideră ca un sistem dinamic cu masă continuuă, rezultând cazul sistemelor cu ∞ grade de libertate dinamice. Prezenţa efortului axial ne conduce la cazul vibraţiilor geometric-neliniare. Sunt studiate cele şase cazuri fundamentale, pentru care sunt determinate pulsaţiile proprii precum şi funcţiile proprii de vibraţie prin integrarea ecuaţiei diferenţiale de ordin IV, astfel suntem conduşi la expresia modurilor normale de vibraţie. Ecuaţia este omogenă şi liniară, iar soluţiile se determină prin aplicarea metodei separării variabilelor temporare de cele spaţiale, astfel suntem conduşi la o problemă mixtă de tip Sturm-Liouville. Prezenţa solicitărilor axiale şi a mediului elastic determină o diminuare a valorilor pulsaţiilor proprii fată de barele necomprimate.

Keywords: geometrically nonlinear vibration, modal analysis, Euler-Bernoulli beam, axial force, Winkler foundation

1. Introduction

In this study, we investigate the dynamic response of elastic beam with axial load (geometrically non-linear vibration) on one parameter elastic linear foundation; also known as a Winkler foundation. Therefore, we take one-span Bernoulli-Euler beam loaded with dynamic loads - distributed forces and concentrate forces. Studies have been done, using the same hypotheses or, the more complex using Timoshenko theory of beams on elastic foundations, considering two different values of Winkler coefficients by Catal S. [1]. Numerical analysis of frequencies values for beams on elastic soil is also done by Oztuk and Coskun in [2].

[*] Corresponding author: Tel./ Fax.: +40-264-401313
E-mail address: tudor.milchis@mecon.utcluj.ro

2. Free vibrations. Normal modes shapes of vibrations

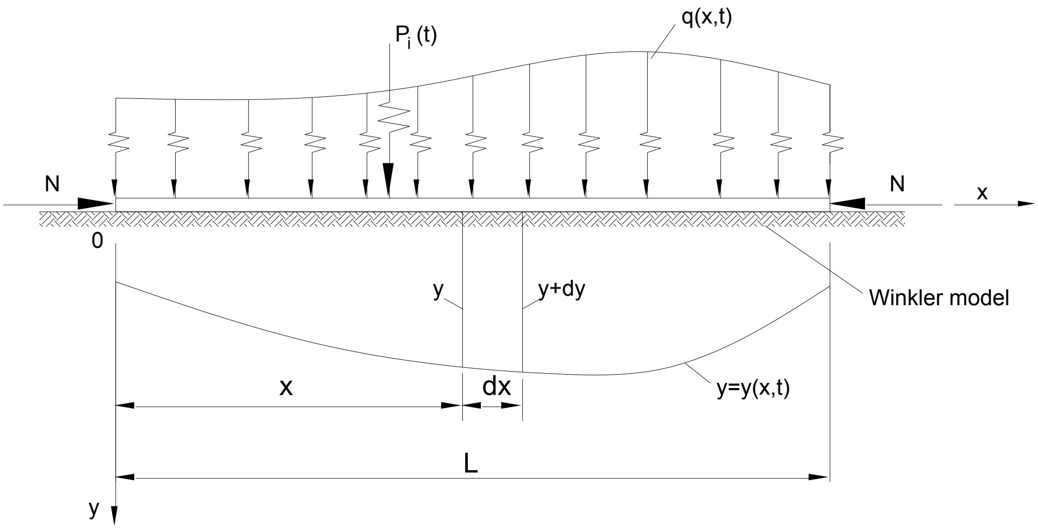

Figure 1. Beam on elastic foundation

The axial load (N), flexural rigidity (EI) and the mass density per unit length (ρ) of the beam are constants. The external load is $q(x, t)$ and $P_i(t)$ for concentrated forces applied in sections x_i. The Winkler constant is k. For the partial derivate equation, will consider a differential length element of beam, considering the external dynamic forces, internal forces and the reaction of foundation in length section x and $x + dx$:

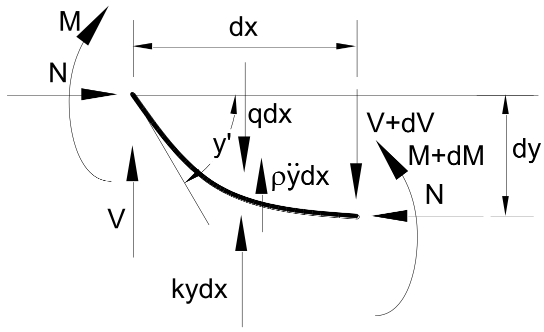

Figure 2. Differential length element of beam

In figure 2, V is the vertical shear force component T. This component, in nonlinear analysis it is tangent at a point x of middle deform fibre, $y = y(x, t)$.
Thus, we can write the equilibrium conditions. The second condition equilibrium is written respect to section "i".

$$\begin{cases} \sum X_i = 0 \\ \sum Y_i = 0 \\ \sum M_i = 0 \end{cases} \Leftrightarrow \begin{cases} N - N = 0 \\ V + dV - V + qdx - \rho\ddot{y}dx - kydx = 0 \\ M + dM - M - Vdx - Ndy + qdx\dfrac{dx}{2} - kydx\dfrac{dx}{2} - \rho\ddot{y}dx\dfrac{dx}{2} = 0 \end{cases} \tag{1}$$

After simplifications of the same or high order differential elements

$$\begin{cases} dV + (q - \rho\ddot{y} - ky)dx = 0 \\ dM - Vdx - Ndy = 0 \end{cases} \tag{2}$$

And further more

$$\begin{cases} \frac{dV}{dx} = -q + \rho\ddot{y} + ky \\ \frac{dM}{dx} = V + Ny' \end{cases} \tag{3}$$

Last relation, after first derivation respect to x:

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} + Ny'' \tag{4}$$

In addition, based on relation (3), we can write

$$\frac{d^2M}{dx^2} = -q + \rho\ddot{y} + ky + Ny'' \tag{5}$$

The differential equation that gives the elastic curve for the deflected beam is

$$M = -\frac{EI}{\rho}; \frac{1}{\rho} \approx y''; M = -EI\, y'' \tag{6}$$

Further

$$-EI\, y'' = -q + \rho\ddot{y} + ky + Ny'' \tag{7}$$

At last, the final expression

$$EI\, y^{IV} + Ny'' + ky + \rho\ddot{y} = q. \tag{8}$$

If external loads are concentrated $P_i(t)$, at sections x_i the Eq. (6) becomes

$$EI\, y^{IV} + Ny'' + ky + \rho\ddot{y} = q + \sum_i P_i\, \delta(x - x_i) \tag{9}$$

where $\delta(x - x_i)$ is the Dirac function.
Comment:
Starting from relation (3)

$$\frac{dM}{dx} = T$$

$$T = V + N\, y' \tag{10}$$

and it reveals the influence of axial load (N) in the value of shear force (T) in non-linear calculus. The partial differential Eq. (7) describes the geometrically non-linear forced vibration of a beam on an elastic foundation.

The partial derivate equation of free vibrations, due to initial conditions (displacements and velocities) without externals loads $q(x, t) = 0$ and $P_i(x) = 0$ is:

$$\mathrm{EI}\, y^{IV} + N\, y'' + ky + \rho\, \ddot{y} = 0 \tag{11}$$

Eq. (11) describes the normal modes shapes of geometrically nonlinear vibrations of o beam resting on elastic foundation. The natural frequencies functions are determined also. This two elements, ω_j and $\phi_j(x)$ determine the normal mode shape "j".

For the integration of Eq. (11), we will apply the variables separation technique. For that the solution of Eq. (11) will be written:

$$y(x,t) = \phi(x) \cdot \eta(t) \tag{12}$$

So further on the Eq. (11) becomes:

$$\mathrm{EI}\, \phi^{IV}(x) \cdot\ \eta(t) + N\, \phi''(x) + \rho\, \phi(x) \cdot \ddot{\eta}(t) = 0 \tag{13}$$

divided by $\rho\phi(x) \cdot \eta(t)$:

$$\frac{\mathrm{EI}}{\rho} \cdot \frac{\phi^{IV}(x)}{\phi(x)} + \frac{N}{\rho} \cdot \frac{\phi''(x)}{\phi(x)} + \frac{k}{\rho} = -\frac{\ddot{\eta}\,(t)}{\eta(t)} = \omega^2 \tag{14}$$

In Eq. (14), the two fractions, of different variables, could not be equal only in the case that both fractions are constants. That constant is ω^2 also known as the natural frequency.

Further, from Eq. (14) we can write the following:

$$\ddot{\eta}(t) + \omega^2 \eta(t) = 0 \tag{15}$$

with the solution

$$\eta(t) = A \cdot \sin(\omega t + \varphi) \tag{16}$$

and it concludes that the free dynamic response is harmonic and dependent of the frequency ω.

The first equality of Eq. (14) is written:

$$\phi^{IV}(x) + n^2 \phi''(x) - \alpha^4 \phi(x) = 0 \tag{17}$$

where

$$n^2 = \frac{N}{\mathrm{EI}}\,; \alpha^4 = \frac{\omega^2 \rho}{\mathrm{EI}} \tag{18}$$

The natural frequencies functions $\omega_j, \phi_j(x)$ equation that results from Eq. (17) are dependent of boundary conditions. The differential equation and the boundary conditions is also known as a Sturm-Liouville problem. We will consider a free end beam at both ends:

$$\begin{cases} \phi^{IV}(x) + n^2 \phi''(x) - \alpha^4 \phi(x) = 0 \\ \phi''(0) = 0 \\ \mathrm{EI} \cdot \phi'''(0) +\ N \cdot \phi'(0) = 0 \\ \phi''(l) = 0 \\ \mathrm{EI} \cdot \phi'''(l) +\ N \cdot \phi'(l) = 0 \end{cases} \tag{19}$$

Solving this problem will determine the normal mode shapes functions $\phi_j(x)$, which are non-trivial solutions and satisfy the Eq. (19) conditions. Thus, the solution of Eq. (17) it is replace by his own solution.

3. The homogeneous solution of differential equation

Eq. (17) is a linear equation, homogeneous, with constant coefficients, which has the particular solutions by the form of $\mathbf{e^{\lambda x}}$. The characteristic equation is:

$$\lambda^4 + n^2\lambda^2 - \alpha^4 = 0 \tag{20}$$

with the solutions

$$\begin{aligned} \lambda_1^2 &= \frac{\sqrt{n^4 + 4\alpha^4} - n^2}{2} > 0 \\ \lambda_2^2 &= -\frac{\sqrt{n^4 + 4\alpha^4} - n^2}{2} < 0 \end{aligned} \tag{21}$$

Thus, the general solution of Eq. (17) is

$$\phi(x) = A \cdot \cosh(\lambda_1 x) + B \cdot \sinh(\lambda_1 x) + C \cdot \cos(\lambda_2 x) + D \cdot \sin(\lambda_2 x) \tag{22}$$

and the constants A, B, C and D will be determined from the boundary conditions of the problem.

4. Examples. The six fundamental cases.

The homogeneous boundary conditions of Sturm-Liouville problem Eq. (19) involve the derivatives of general solution from Eq. (22):

$$\begin{aligned} \phi'(x) &= A\lambda_1 \cdot \sinh(\lambda_1 x) + B\lambda_1 \cdot \cosh(\lambda_1 x) - C\lambda_2 \cdot \sin(\lambda_2 x) + D\lambda_2 \cdot \cos(\lambda_2 x) \\ \phi''(x) &= A\lambda_1^2 \cdot \cosh(\lambda_1 x) + B\lambda_1^2 \cdot \sinh(\lambda_1 x) - C\lambda_2^2 \cdot \cos(\lambda_2 x) - D\lambda_2^2 \cdot \sin(\lambda_2 x) \\ \phi'''(x) &= A\lambda_1^3 \cdot \sinh(\lambda_1 x) + B\lambda_1^3 \cdot \cosh(\lambda_1 x) + C\lambda_2^3 \cdot \sin(\lambda_2 x) - D\lambda_2^3 \cdot \cos(\lambda_2 x) \end{aligned} \tag{23}$$

4.1 Simply supported beam

According to Eq. (19), the differential equation is replaced by his own solution. The boundary conditions results from displacements and bending moment null values Thus, the Sturm-Liouville problem is:

$$\begin{cases} \phi(x) = A \cdot \cosh(\lambda_1 x) + B \cdot \sinh(\lambda_1 x) + C \cdot \cos(\lambda_2 x) + D \cdot \sin(\lambda_2 x) \\ \phi(0) = 0 \\ \phi''(0) = 0 \\ \phi\,(l) = 0 \\ \phi''(l) = 0 \end{cases} \tag{24}$$

Thus, the system is:

$$\begin{cases} A + C = 0 \\ A\lambda_1^2 - C\lambda_2^2 = 0 \\ A \cdot \cosh(\lambda_1 l) + B \cdot \sinh(\lambda_1 l) + C \cdot \cos(\lambda_2 l) + D \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1^2 \cdot \cosh(\lambda_1 l) + B\lambda_1^2 \cdot \sinh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \end{cases} \tag{25}$$

It is a linear equation system, homogeneous. To avoid a trivial solution, at least one of the integrations constants must be different to zero, we must condition the determinant to be equal to zero. Therefore:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1^2 & 0 & -\lambda_1^2 & 0 \\ \cosh(\lambda_1 l) & \sinh(\lambda_1 l) & \cos(\lambda_2 l) & \sin(\lambda_2 l) \\ \lambda_1^2\cosh(\lambda_1 l) & \lambda_1^2\sinh(\lambda_1 l) & -\lambda_2^2\cosh(\lambda_2 l) & -\lambda_2^2\sin(\lambda_2 l) \end{vmatrix} = 0 \tag{26}$$

Only that, the system Eq. (25) can be reduced to a two-equation system with two variables. The first two equations, with constants A, C can be separated taking notice that $\lambda_1 \neq \lambda_2$, thus $A = C = 0$. The last two relations from Eq. (25) can be written:

$$\begin{cases} B \cdot \sinh(\lambda_1 l) + D \cdot \sin(\lambda_2 l) = 0 \\ B\lambda_1^2 \cdot \sinh(\lambda_1 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \end{cases} \tag{27}$$

this system of homogeneous equations is with two unknown variables $B \cdot \sinh(\lambda_1 l)$ and $D \cdot \sin(\lambda_2 l)$. Taking notice that $\lambda_1 \neq \lambda_2$, so we can write:

$$B \cdot \sinh(\lambda_1 l) = 0\,; \quad D \cdot \sin(\lambda_2 l) = 0. \tag{28}$$

But $\sinh(\lambda_1 l) \neq 0$ thus $B = 0$, farther more for non trivial solution of Eq. (22), $\phi(x) \neq 0$, and the integration variable $D \neq 0$, results:

$$\sinh(\lambda_2 l) = 0 \tag{29}$$

The Eq. (29) is also known as the frequency equation and D is an arbitrary constant. The mode shape functions are:

$$\phi_j(x) = \sin(\lambda_2 x) \tag{30}$$

From Eq. (29) results the natural frequencies values:

$$\lambda_2 l = j\pi; j = 1, 2, 3, \ldots \tag{31}$$

$$\lambda_2 = \frac{j\pi}{l};\ \lambda_2^2 = \frac{j^2\pi^2}{l^2} \tag{32}$$

From Eq. (21), with the absolute value of $|\lambda_2^2|$, results:

$$\frac{\sqrt{n^4 + 4\alpha^4} + n^2}{2} = \frac{j^2\pi^2}{l^2} \tag{33}$$

To obtain natural frequencies values ω_j, also α_j^4, will proceed further:

$$\frac{\sqrt{n^4 + 4\alpha^4}}{2} = \frac{j^2\pi^2}{l^2} - \frac{n^2}{2}$$

$$\frac{n^4 + 4\alpha^4}{4} = \frac{j^4\pi^4}{l^4} - \frac{j^2\pi^4 n^2}{l^2} + \frac{n^4}{4}$$

$$\alpha_j^4 = \frac{j^4\pi^4}{l^4}\left(1 - \frac{n^2 l^2}{j^2\pi^2}\right) \tag{34}$$

As a result Eq. (18) can be written as follows, without any comments

$$\frac{\omega_j^2\rho - k}{EI} = \frac{j^4\pi^4}{l^4}\left(1 - \frac{n^2 l^2}{j^2\pi^2}\right) \tag{35}$$

thus,

$$\omega_j^2 = \frac{EI}{\rho}\left[\frac{j^4\pi^4}{l^4}\left(1 - \frac{n^2 l^2}{j^2\pi^2}\right) + \frac{k}{EI}\right] \tag{36}$$

Comments

- in normal cases of elastic beams (with no axial load $n = 0$; without resting on elastic foundation $k = 0$)

$$\omega_j^2 = \frac{EI}{\rho}\frac{j^4\pi^4}{l^4} \tag{37}$$

- with axial load present

$$\omega_j^2 = \frac{EI}{\rho}\frac{j^4\pi^4}{l^4}\left(1 - \frac{n^2 l^2}{j^2\pi^2}\right) \tag{38}$$

- the natural frequencies values for the beam with axial load $(N \neq 0)$ are smaller than frequencies of a beam without axial load. That occurs in reducing the rigidity of the beam, due the axial load.
- the beam is resting on elastic foundation is more rigid, therefore the natural frequencies values are higher Eq. (36)
- the axial load and the elastic foundation don`t modify the shape of mode functions $\phi_j(x)$ (trigonometrically function sinus is still the prime function); only the rate value and ordinate are modify

4.2 Free beam at both ends

Boundary conditions for a free beam at the bought ends, considering that the bending moments and shear forces are zero $(M = 0; T = 0)$, are:

$$\begin{cases} M(0) = 0 \\ T(0) = 0 \\ M(l) = 0 \\ T(l) = 0 \end{cases} \Rightarrow \begin{cases} \phi''(0) = 0 \\ EI \cdot \phi'''(0) + N \cdot \phi'(0) == 0 \\ \phi''(l) = 0 \\ EI \cdot \phi'''(l) + N \cdot \phi'(l) = 0 \end{cases} \tag{39}$$

As a result, Eq. (23) can be written as follows, and we obtained a linear homogenous system of equations:

$$\begin{cases} A\lambda_1^2 - C\lambda_2^2 = 0 \\ B\lambda_1^3 - D\lambda_2^3 + n^2(B\lambda_1 - D\lambda_2) = 0 \\ A\lambda_1^2 \cdot ch(\lambda_1 l) + B\lambda_1^2 \cdot sh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1^3 \cdot sh(\lambda_1 l) + B\lambda_1^3 \cdot ch(\lambda_1 l) + C\lambda_2^3 \cdot \sin(\lambda_2 l) - D\lambda_2^3 \cdot \cos(\lambda_2 l) + \\ +n^2[A\lambda_1 \cdot sh(\lambda_1 l) + B\lambda_1 \cdot ch(\lambda_1 l) - C\lambda_2 \cdot \sin(\lambda_2 l) + D\lambda_2 \cdot \cos(\lambda_2 l)] = 0 \end{cases} \tag{40}$$

To avoid trivial solution, we must condition the determinant to be equal to zero. Therefore:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{41}$$

Where

$$a_{11} = \lambda_1^2 \cdot [\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]$$
$$a_{12} = \lambda_1 \cdot \left[\lambda_1 \mathrm{sh}(\lambda_1 l) - \lambda_2 \cdot \frac{\lambda_1^2 + n^2}{n^2 - \lambda_1^2} \sin(\lambda_2 l)\right]$$
$$a_{21} = \lambda_1 \cdot \left[(\lambda_1^2 + n^2)\mathrm{sh}(\lambda_1 l) + \frac{\lambda_1}{\lambda_2}(\lambda_1^2 - n^2)\sin(\lambda_2 l)\right]$$
$$a_{22} = \lambda_1(\lambda_1^2 + n^2)[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]$$

which represents the frequencies equation of a beam with axial load, resting on elastic foundation, with free ends.
Solving this determinant Eq. (41), it results:

$$2\lambda_1^3\lambda_2^3[1 - \mathrm{ch}(\lambda_1 l)\cos(\lambda_1 l)] + (\lambda_2^6 - \lambda_1^6)\mathrm{sh}(\lambda_1 l)\sin(\lambda_1 l) = 0 \tag{42}$$

And notating

$$(\lambda_1 l)^2 = \frac{1}{2}\left(\sqrt{(nl)^4 + 4(\alpha l)^4} - (nl)^2\right)$$
$$(\lambda_2 l)^2 = \frac{1}{2}\left(\sqrt{(nl)^4 + 4(\alpha l)^4} + (nl)^2\right) \tag{43}$$

$$(\lambda_2 l)^2 - (\lambda_1 l)^2 = (nl)^2; nl = \nu \tag{44}$$

Further, with the notation $z = (\lambda_2 l)$; the primary unknown variable and $(\lambda_1 l)^2 = z^2 - \nu^2$
Thus, Eq. (42) becomes

$$2\left(\sqrt{z^2 - \nu^2}\right)^3 \cdot z^3 \cdot \left[1 - \mathrm{ch}\left(\sqrt{z^2 - \nu^2}\right)\cos(z)\right] + \\ + \left[z^6 - \left(\sqrt{z^2 - \nu^2}\right)^6\right]\mathrm{sh}\left(\sqrt{z^2 - \nu^2}\right)\sin(z) = 0 \tag{45}$$

This equation is also known as the frequency equation in the z variable. The solutions of these equations are obtained for explicit values of compression factor ν. Thus, it is obtained the solutions:

$$0 < z_1 < z_2 < z_3 < \cdots$$
$$0 < (\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots (\lambda_2 l)_j; j = 1,2,3, \ldots$$

From

$$(\lambda_2 l)_j^2 = \frac{1}{2}\left[\sqrt{(nl)^4 + 4(\alpha l)^4} + (nl)^2\right] = \frac{1}{2}\left[\sqrt{\nu^4 + 4(\alpha l)^4} + \nu^2\right] \tag{46}$$

results

$$\frac{\sqrt{\nu^4 + 4(\alpha l)^4}}{2} = (\lambda_2 l)_j^2 - \frac{\nu^2}{2}; \quad \frac{\nu^4 + 4(\alpha l)^4}{4} = (\lambda_2 l)_j^4 - (\lambda_2 l)_j^2\nu^2 + \frac{\nu^4}{4}$$
$$(\alpha l)_j^4 = (\lambda_2 l)_j^4\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)^2\right] \tag{47}$$

$$\alpha_j^4 = \lambda_2^4\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)^2\right] \tag{48}$$

thus

$$\frac{\omega_j^2\rho - k}{EI} = \lambda_2^4\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)^2\right] \tag{49}$$

$$\frac{\omega_j^2\rho}{EI} = \lambda_2^4\left[1-\left(\frac{\nu}{\lambda_2 l}\right)^2\right]+\frac{k}{EI} \tag{50}$$

$$\omega_j^2 = \frac{EI}{\rho}\left(\lambda_2^4\left[1-\left(\frac{\nu}{\lambda_2 l}\right)^2\right]+\frac{k}{EI}\right) \tag{51}$$

Mode shape functions of vibration and the constants of integration will be formulate according to only one constant of integration, assign with an arbitrary value.
From Eq. (40) results

$$C = \frac{\lambda_1^2}{\lambda_2^2}\,A$$
$$D = \frac{\lambda_1}{\lambda_2}\cdot\frac{{\lambda_1}^2+\nu^2}{{\lambda_2}^2-\nu^2}\cdot B \tag{52}$$

and further, from Eq. (40) and in the end:

$$B = -\frac{(\lambda_2 l)[\cosh(\lambda_1 l)-\cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_2 l)-(\lambda_1 l)\sin(\lambda_2 l)}\cdot A = -k_j A$$
$$C = \frac{(\lambda_1 l)^2}{(\lambda_2 l)^2}A \tag{53}$$
$$D = -\frac{(\lambda_1 l)^3}{(\lambda_2 l)^3}\frac{(\lambda_2 l)[\cosh(\lambda_1 l)-\cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_2 l)-(\lambda_1 l)\sin(\lambda_2 l)}\cdot A = -\frac{(\lambda_1 l)^3}{(\lambda_2 l)^3}\,k_j A$$

The variables and constants from Eq. (40) have the index number "j" A_j, B_j, C_j and D_j so we can concludeor, with the assign

$$k_j = \frac{(\lambda_1 l)[\mathrm{ch}(\lambda_1 l)-\cos(\lambda_2 l)]}{(\lambda_1 l)\mathrm{sh}(\lambda_1 l)-(\lambda_2 l)\dfrac{(\lambda_1 l)^2+\nu^2}{(\lambda_2 l)^2-\nu^2}\sin(\lambda_2 l)}$$

$$\phi_j(x) = \mathrm{ch}\left[(\lambda_1 l)_j\frac{x}{L}\right]-k_j\mathrm{sh}\left[(\lambda_1 l)_j\frac{x}{L}\right]+\left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j^2\cos\left[(\lambda_2 l)_j\frac{x}{l}\right]- \\ -\left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j\sin\left[(\lambda_2 l)_j\frac{x}{l}\right] \tag{54}$$

4.3 Cantilevered beam

Boundary conditions for a cantilevered beam, with the clamping at the left end, are:

$$\begin{cases}\phi(0)=0\\ \phi'(0)=0\\ \phi''(l)=0\\ EI\cdot\psi'''(l)+N\cdot\psi'(l)=0\end{cases} \tag{55}$$

So, from Eq. (19) and Eq. (20)

$$\begin{cases}A+C=0\\ B\lambda_1+D\lambda_2=0\\ A\lambda_1^2\cdot\mathrm{ch}(\lambda_1 l)+B\lambda_1^2\cdot\mathrm{sh}(\lambda_1 l)-C\lambda_2^2\cdot\cos(\lambda_2 l)-D\lambda_2^2\cdot\sin(\lambda_2 l)=0\\ A\lambda_1(\lambda_1^2+n^2)\cdot\mathrm{sh}(\lambda_1 l)+B\lambda_1(\lambda_1^2+n^2)\cdot\mathrm{ch}(\lambda_1 l)-C\lambda_1(\lambda_1^2+n^2)\cdot\sin(\lambda_2 l)-\end{cases} \tag{56}$$

$$-D\lambda_1(\lambda_1^2 + n^2)\cdot\cos(\lambda_2 l) = 0$$

It is easy to show, from the first two equations, that

$$\begin{cases} C = -A \\ D\lambda_2 = B\lambda_1 \end{cases};\ D = -\frac{\lambda_1}{\lambda_2}B$$

and further

$$\begin{cases} A[\lambda_1^2\,\mathrm{ch}(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l)] + B\lambda_1[\lambda_1\,\mathrm{sh}(\lambda_1 l) + \lambda_2\sin(\lambda_2 l)] = 0 \\ A[\lambda_1^3\,\mathrm{sh}(\lambda_1 l) - \lambda_2^3\sin(\lambda_2 l) + n^2\lambda_1\mathrm{sh}(\lambda_1 l) + n^2\lambda_2\sin(\lambda_2 l)] + \end{cases} \tag{57}$$

$$+B\lambda_1[\lambda_1^2\mathrm{ch}(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l) + n^2\mathrm{ch}(\lambda_1 l) - n^2\cos(\lambda_2 l)] = 0$$

For nontrivial solution, we must condition the determinant to be equal to zero. Therefore:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

$$\begin{aligned} a_{11} &= \lambda_1^2\mathrm{ch}(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l) \\ a_{12} &= \lambda_1\,\mathrm{sh}(\lambda_1 l) + \lambda_2\sin(\lambda_2 l) \\ a_{21} &= \lambda_1^3\,\mathrm{sh}(\lambda_1 l) - \lambda_2^3\sin(\lambda_2 l) + n^2[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)] \\ a_{22} &= \lambda_1^2\mathrm{ch}(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l) + n^2[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)] \end{aligned} \tag{58}$$

The Eq. (62) represents the frequencies equation. Further more, evaluating the above determinant it results, with the notations

$$nl = \nu;\quad \lambda_2 l = z;\quad \lambda_1 l = \sqrt{z^2 - n^2}$$

$$\begin{aligned} &(\lambda_1 l)^4 + (\lambda_2 l)^4 + (nl)^2[(\lambda_1 l)^2 - (\lambda_2 l)^2] + \\ &+\big[2(\lambda_1 l)^2(\lambda_2 l)^2 + (nl)^2[(\lambda_2 l)^2 - (\lambda_1 l)^2]\big]\,\mathrm{ch}(\lambda_1 l)\cos(\lambda_2 l) + \\ &+(\lambda_1 l)(\lambda_2 l)[(\lambda_2 l)^2 - (\lambda_1 l)^2 - 2(nl)^2]\,\mathrm{sh}(\lambda_1 l)\sin(\lambda_2 l) = 0 \end{aligned} \tag{59}$$

Solutions from Eq. (64)

$$z_1 < z_2 < z_3 < \cdots < z_j < \cdots$$

$$(\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots < (\lambda_2 l)_j < \cdots$$

$$(\lambda_2 l)_j^2 = \frac{1}{2}\left[\sqrt{(nl)^4 + 4(\alpha l)^4} + (nl)^2\right] = \frac{1}{2}\left[\sqrt{\nu^4 + 4(\alpha l)^4} + \nu^2\right] \tag{60}$$

$$\frac{\sqrt{\nu^4 + 4(\alpha l)_j^4}}{2} = (\lambda_2 l)_j^2 - \frac{\nu^2}{2};\quad (\alpha l)_j^4 = (\lambda_2 l)_j^4 - \nu^2(\lambda_2 l)_j^2$$

$$(\alpha l)_j^4 = (\lambda_2 l)_j^4\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)_j^2\right];\quad \alpha_j^4 = \frac{(\lambda_2 l)_j^4}{l^4}\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)_j^2\right]$$

$$\frac{\omega_j^2\rho - k}{EI} = \frac{(\lambda_2 l)_j^4}{l^4}\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)_j^2\right]$$

$$\frac{\omega_j^2\rho}{EI} = \frac{(\lambda_2 l)_j^4}{l^4}\left[1 - \left(\frac{\nu}{\lambda_2 l}\right)_j^2\right] + \frac{k}{EI}$$

$$\omega_j^2 = \frac{EI}{\rho l^4}\left((\lambda_2 l)_j^4\left[1 - \frac{\nu^2}{(\lambda_2 l)_j^2}\right] + \frac{k}{EI}\right) \tag{61}$$

The normal shapes of vibrations functions $\phi_j(x)$ to be determined, the constants of integration

A, B, C and D, dependent to each other, we must assign an arbitrary value to one of them $A = 1$ and the remain constants are easy to find.
From Eq. (61) and Eq. (62):

$$C = -A$$
$$\frac{B}{A} = -\frac{\lambda_1^2 \cosh(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l)}{\lambda_1^2 \sinh(\lambda_1 l) + \lambda_1\lambda_2 \sin(\lambda_2 l)}$$

$$B = -\frac{(\lambda_1 l)^2 \cosh(\lambda_1 l) + (\lambda_2 l)^2 \cos(\lambda_2 l)}{(\lambda_1 l)^2 \sinh(\lambda_1 l) + (\lambda_1 l)(\lambda_2 l)\sin(\lambda_2 l)} \cdot A = -k_j A \tag{62}$$

$$D = -\frac{\lambda_1 l}{\lambda_2 l} B = \frac{\lambda_1 l}{\lambda_2 l} k_j A,$$
$$k_j = \frac{(\lambda_1 l)^2 \operatorname{ch}(\lambda_1 l) + (\lambda_2 l)^2 \cos(\lambda_2 l)}{(\lambda_1 l)^2 \operatorname{sh}(\lambda_1 l) + (\lambda_1 l)(\lambda_2 l)\sin(\lambda_2 l)}$$

Therefore, the final normal shapes form expression:

$$\phi_j(x) = \operatorname{ch}\left[(\lambda_1 l)_j \frac{x}{l}\right] - k_j \operatorname{sh}\left[(\lambda_1 l)_j \frac{x}{l}\right] - \cos\left[(\lambda_2 l)_j \frac{x}{l}\right] + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j \sin\left[(\lambda_2 l)_j \frac{x}{l}\right] \tag{63}$$

4.4 Double clamped beam

Boundary conditions at the ends of beam $x = 0; x = l$ are:

$$\begin{cases} \phi(0) = 0 \\ \phi'(0) = 0 \\ \phi(l) = 0 \\ \phi'(l) = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} A - C = 0 \\ B\lambda_1 - D\lambda_2 = 0 \\ A\cosh(\lambda_1 l) + B\sinh(\lambda_1 l) + C\cos(\lambda_2 l) + D\sin(\lambda_2 l) = 0 \\ A\,\lambda_1 \sinh(\lambda_1 l) + B\lambda_1 \cosh(\lambda_1 l) - C\lambda_2 \sin(\lambda_2 l) - D\lambda_2 \cos(\lambda_2 l) = 0 \end{cases} \tag{64}$$

with the remark

$$\begin{cases} C = -A \\ D\lambda_2 = -B\lambda_1 \end{cases}; D = -\frac{\lambda_1}{\lambda_2} B$$

$$\begin{cases} A\lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] + B[\lambda_2 \sinh(\lambda_1 l) - \lambda_1 \sin(\lambda_2 l)] = 0 \\ A[\lambda_1 \sinh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l)] + B\lambda_1[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] = 0 \end{cases} \tag{65}$$

$$\begin{vmatrix} \lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] & \lambda_2 \sinh(\lambda_1 l) - \lambda_1 \sin(\lambda_2 l) \\ \lambda_1 \sinh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l) & \lambda_1[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] \end{vmatrix} = 0 \tag{66}$$

$$\lambda_1\lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]^2 - [\lambda_1 \sinh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l)][\lambda_2 \sinh(\lambda_1 l) - \lambda_1 \sin(\lambda_2 l)] = 0$$
$$\lambda_1\lambda_2 \cosh^2(\lambda_1 l) - 2\lambda_1\lambda_2 \cosh(\lambda_1 l)\cos(\lambda_2 l) + \lambda_1\lambda_2 \cos^2(\lambda_2 l) - \lambda_1\lambda_2 \sinh^2(\lambda_1 l)$$
$$+ \lambda_1^2 \sinh(\lambda_1 l)\sin(\lambda_2 l) - \lambda_2^2 \sinh(\lambda_1 l)\sin(\lambda_2 l) + \lambda_1\lambda_2 \sin^2(\lambda_2 l) = 0$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_2 - 2\lambda_1\lambda_2 \cosh(\lambda_1 l)\cos(\lambda_2 l) - (\lambda_2^2 - \lambda_1^2)\sinh(\lambda_1 l)\sin(\lambda_2 l) = 0$$
$$2\lambda_1\lambda_2[1 - \cosh(\lambda_1 l)\cos(\lambda_2 l)] - (\lambda_2^2 - \lambda_1^2)\sinh(\lambda_1 l)\sin(\lambda_2 l) = 0$$
$$1 - \cosh(\lambda_1 l)\cos(\lambda_2 l) - \frac{\lambda_2^2 - \lambda_1^2}{2\lambda_1\lambda_2}\sinh(\lambda_1 l)\sin(\lambda_2 l) = 0$$

$$1 - \cosh(\lambda_1 l)\cos(\lambda_2 l) - \frac{(\lambda_2 l)^2 - (\lambda_1 l)^2}{2(\lambda_1 l)(\lambda_2 l)}\sinh(\lambda_1 l)\sin(\lambda_2 l) = 0 \tag{67}$$

$$(\lambda_2 l)^2 - (\lambda_1 l)^2 = \nu^2; \quad (\lambda_1 l) = \sqrt{z^2 - \nu^2}$$

$$1 - \cosh\left(\sqrt{z^2-\nu^2}\right)\cos(z) - \frac{\nu^2}{2z\sqrt{z^2-\nu^2}}\sinh\left(\sqrt{z^2-\nu^2}\right)\sin(z) = 0 \tag{68}$$

The solutions are

$$0 < z_1 < z_2 < z_3 < \cdots < z_j < \cdots$$
$$0 < (\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots < (\lambda_2 l)_j < \cdots$$

the natural frequencies ω_j values are determined with Eq. (66).
For mode shapes functions, the integration constants are:

$$C = -A$$

$$\frac{B}{A} = -\frac{\lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{\lambda_2 \sinh(\lambda_1 l) - \lambda_1 \sin(\lambda_2 l)}; \quad B = -\frac{(\lambda_2 l)[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_1 l) - (\lambda_1 l)\sin(\lambda_2 l)} \cdot A = -k_j A$$

$$D = -\frac{(\lambda_1 l)}{(\lambda_2 l)} B = \frac{(\lambda_1 l)}{(\lambda_2 l)} k_j A$$

where the final expression of normal shapes of vibration

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \cos(\lambda_1 l)_j \cdot \frac{x}{l} - \cos(\lambda_2 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \tag{69}$$

4.5 Clamped – simply supported beam

In this case, the boundary conditions are:

$$\begin{cases} \phi(0) = 0 \\ \phi'(0) = 0 \\ \phi(l) = 0 \\ \phi''(l) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A - C = 0 \\ B\lambda_1 - D\lambda_2 = 0 \\ A\cosh(\lambda_1 l) + B\sinh(\lambda_1 l) + C\cos(\lambda_2 l) + D\sin(\lambda_2 l) = 0 \\ A\lambda_1^2\cosh(\lambda_1 l) + B\lambda_1^2\sinh(\lambda_1 l) - C\lambda_2^2\cos(\lambda_2 l) - D\lambda_2^2\sin(\lambda_2 l) = 0 \end{cases} \tag{70}$$

$$\begin{cases} C = -A \\ D\lambda_2 = -B\lambda_1 \end{cases}; D = -\frac{\lambda_1}{\lambda_2}B$$

$$\begin{cases} A\lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] + B[\lambda_2\sinh(\lambda_1 l) - \lambda_1\sin(\lambda_2 l)] = 0 \\ A[\lambda_1^2\cosh(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l)] + B\lambda_1[\lambda_1\sinh(\lambda_1 l) + \lambda_2\sin(\lambda_2 l)] = 0 \end{cases} \tag{71}$$

$$\begin{vmatrix} \lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)] & \lambda_2\sinh(\lambda_1 l) - \lambda_1\sin(\lambda_2 l) \\ \lambda_1^2\cosh(\lambda_1 l) + \lambda_2^2\cos(\lambda_2 l) & \lambda_1[\lambda_1\sinh(\lambda_1 l) + \lambda_2\sin(\lambda_2 l)] \end{vmatrix} = 0 \tag{72}$$

$$\lambda_2 l = z; \quad \lambda_1 l = \sqrt{z^2 - \nu^2}$$

$$\sqrt{z^2-\nu^2}\tan(z) - z\tanh\left(\sqrt{z^2-\nu^2}\right) - \nu^2 = 0 \tag{73}$$

The solutions are

$$0 < z_1 < z_2 < z_3 < \cdots < z_j < \cdots$$
$$0 < (\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots < (\lambda_2 l)_j < \cdots$$

the natural frequencies ω_j values are determined with Eq. (46),

$$C = -A$$

$$\frac{B}{A} = -\frac{\lambda_2[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{\lambda_2 \sinh(\lambda_1 l) - \lambda_1 \sin(\lambda_2 l)}; \quad B = -\frac{(\lambda_2 l)[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_1 l) - (\lambda_1 l)\sin(\lambda_2 l)} \cdot A = -k_j A$$

$$D = -\frac{\lambda_1}{\lambda_2} B = -\left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j \cdot (-k_j) A$$

and normal shapes of vibration

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \cos(\lambda_1 l)_j \cdot \frac{x}{l} - \cos(\lambda_2 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \tag{74}$$

4.6 Free bearing- free at the other end

$$\begin{cases} \phi(0) = 0 \\ \phi''(0) = 0 \\ \phi''(l) = 0 \\ EI \cdot F'''(l) + N \cdot F'(l) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A + C = 0 \\ A\lambda_1^2 - C\lambda_2^2 = 0 \\ A\lambda_1^2\,\mathrm{sh}(\lambda_1 l) + B\lambda_1^2\,\mathrm{ch}(\lambda_1 l) - C\lambda_2^2\cos(\lambda_2 l) - D\,\lambda_2^2\sin(\lambda_2 l) = 0 \\ A\,\lambda_1^3\mathrm{sh}(\lambda_1 l) + B\,\lambda_1^3\mathrm{ch}(\lambda_1 l) + C\lambda_2^3\sin(\lambda_2 l) - D\lambda_2^3\cos(\lambda_2 l) + \\ +n^2[A\,\lambda_1\mathrm{sh}(\lambda_1 l) + B\,\lambda_1\mathrm{ch}(\lambda_1 l) - C\lambda_2\sin(\lambda_2 l) + D\lambda_2\cos(\lambda_2 l)] = 0 \end{cases} \tag{75}$$

$$\begin{cases} A + C = 0 \\ A\lambda_1^2 - C\lambda_2^2 = 0 \end{cases}; \quad A = C = 0$$

$$\begin{cases} B\lambda_1^2\,\mathrm{sh}(\lambda_1 l) - D\,\lambda_2^2\sin(\lambda_2 l) = 0 \\ B\lambda_1\mathrm{ch}(\lambda_1 l)(\lambda_1^2 + n^2) - D\lambda_2\cos(\lambda_2 l)\,(\lambda_2^2 - n^2) = 0 \end{cases} \tag{76}$$

$$\begin{vmatrix} \lambda_1^2\,\mathrm{sh}(\lambda_1 l) & -\lambda_2^2\sin(\lambda_2 l) \\ \lambda_1\mathrm{ch}(\lambda_1 l)(\lambda_1^2 + n^2) & -\lambda_2\cos(\lambda_2 l)\,(\lambda_2^2 - n^2) \end{vmatrix} = 0 \tag{77}$$

With the notation $z = \lambda_2 l$

$$(z^2 - \nu^2)\sqrt{z^2 - \nu^2}\,\mathrm{sh}\left(\sqrt{z^2 - \nu^2}\right)\cos(z) - z^3\mathrm{ch}\left(\sqrt{z^2 - \nu^2}\right)\sin(z) = 0 \tag{78}$$

with the solutions

$$0 < z_1 < z_2 < z_3 < \cdots < z_j < \cdots$$

$$0 < (\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots < (\lambda_2 l)_j < \cdots$$

The normal shades functions

$$A = C = 0$$

$$\frac{D}{B} = \frac{\lambda_1^2 \sinh(\lambda_1 l)}{\lambda_2^2 \sin(\lambda_2 l)}; \quad D = \frac{(\lambda_1 l)^2 \sinh(\lambda_1 l)}{(\lambda_2 l)^2 \sin(\lambda_2 l)} \cdot B = k_j B$$

$$k_j = \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j^2 \frac{\sinh(\lambda_1 l)}{\sin(\lambda_2 l)}; \quad B = 1$$

$$\phi_j(x) = \mathrm{sh}\left[(\lambda_1 l)_j \frac{x}{l}\right] + k_j \sin\left[(\lambda_2 l)_j \frac{x}{l}\right] \tag{79}$$

5. Numerical analysis

The frequencies values of first mode shape for the six fundamental cases of an beam resting on elastic foundation are determined, considering a rectangular beam of 25 cm width and 45 cm height, Poisson`s ratio 0.2, Young modulus 30 GPa and 6.00 m length. The numerical models were created using Abaqus [5] commercial soft, using 2D finite elements type B21. The external load was considered for three sets of external axial load. The compressions coefficient ν has the value 0, 1 and 2. When the beam is clamped on one end and free at the other only the first two values were analyzed. The results are showed in Table 1.

Table 1. Frequencies values

Case	ν	Frequencies (analytical mode) $[s^{-1}]$	Frequencies (Abaqus) $[s^{-1}]$	Differences [%]
Simply supported	0	68.645	62.985	8.98
	1	67.370	62.680	7.48
	2	63.390	61.753	2.65
Free at both ends	0	68.623	64.710	6.047
	1	57.533	64.660	11.022
	2	56.033	62.458	13.071
Cantilever beam	0	64.721	66.527	2.714
	1	63.976	66.442	3.712
Double clamped beam	0	78.623	76.713	2.490
	1	72.622	76.442	4.997
	2	66.555	75.535	11.888
Clamped-simply supported	0	74.727	70.257	6.363
	1	73.523	69.936	5.129
	2	64.789	68.960	6.049
Free bearing-free at the other end	0	67.727	65.114	4.013
	1	62.846	65.007	3.325
	2	60.458	64.458	6.477

6. Conclusions

Free nongeometrical traverse vibrations are studied in the present paper. Normal shape modes of beams on elastic foundations, considering Winkler`s model. Considering six cases of a beam, resting on elastic foundations, the natural frequencies and normal mode shapes functions were determined. The values of natural frequencies decrees when the axial load is increasing due to the rigidity loss.

Percentage differences are showed. Due to the complexity of mathematical equations, the maximum difference in the values determined using the proposed method and the numerical finite element method is 11.88%.

Nonlinear geometrically vibrations problems of beams are difficult to analyze with existing commercial softs.

7. References

[1] Catal S., Demirdag O. *Solution of eigenvalue problem of timoshenko beam on elastic foundation by differential transform method and transfer matrix method*, Journal of Science Technology, vol 2, pp 49-70, 2012.

[2] Ozturk B., Coskun S.B. *Analytical Solution for Free Vibration Analysis of Beam on Elastic Foundation with Different Support Condition*. Mathematical Problems in Engineering, vol 2013, Hindawi Publications Corporation.

[3] Bârsan G.M. *Dinamica şi stabilitatea construcţiilor*. Bucureşti: Editura didactică şi pedagogică, 1979.

[4] Ifrim M. *Dinamica structurilor şi inginerie seismică*. Bucureşti: Editura didactică şi pedagogică, 1984.

[5] Ionescu C. *Stabilitatea şi dinamica construcţiilor*. Iaşi: Editura Societăţii Academice Matei-Teiu Botez, 2007

[6] Nowacki W. *Dinamica sistemelor elastice*. Bucureşti: Editura tehnică, 1969.

[7] ABAQUS user Manual, Version 6.6.