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Nonlinear Vibrations of Elastic Beams

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Abstract

In this paper is studied the free vibrations of beams with axial load (nonlinear geometrical vibrations). The elastic beam is considered to have continuous mass. This problem can be included into ∞ dynamical degree of freedom systems. The mode shapes functions and natural frequencies are determined based on forth order linear and homogenous differential equation of vibrations Eq. (8). The solutions of that equation are determined by the separations of variables method Eq. (9), thus we are lead to a Sturm-Lioville problem. It is observed a decrease of natural frequencies values, which corresponds to a decrease rigidity of compressed beams, due to uncompressed ones. The six fundamental cases are highlighted. In every case the natural frequencies values and the modal shapes functions are determined, witch define the modal shapes of vibrations.

Rezumat

În lucrare se studiază vibraţiile libere ale grinzilor comprimate (vibraţii geometric neliniare). Grinda este considerată ca un sistem dinamic cu masă continuuă, astfel ne situăm în cazul sistemelor cu ∞ grade de libertate dinamice. Funcţiile formelor proprii de vibraţie precum şi pulsaţiile proprii sunt determinate integrând ecuaţia diferenţială de ordin IV, liniară şi omogenă (8). Soluţiile acestei ecuaţii se determină aplicând metoda separării variabilelor (9) fiind conduşi la o problem Sturm-Liouville. Se constată o diminuare a pulsaţiilor proprii, care corespunde cu o reducere a rigidităţii barelor comprimate, în raport cu cele necomprimate. Sunt prezentate cele şase cazuri fundamentale. Pentru fiecare caz sunt prezentate pulsaţiile proprii precum şi funcţiile formelor proprii de vibraţie , care definesc modurile normale de vibraţie.

Keywords: partial differential equation, geometrically nonlinear vibration, compression factor, Euler-Bernoulli beam, axial force.

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1. Introduction

Beams vibrations subject with axial load represents a special part of the large engineering domain. Structures that are representative to this field are tall building, communications towers or above ground water towers. Those structures are loaded, beside the axial load from the gravity field, with cross forces due to wind load or earthquake. The traverse vibrations occur in these conditions. Several researchers have dedicated time to free vibrating beam with axial load present. During period of 1960-1980 the studies have been concentrated to the analytical solutions of the mathematical problems. After the 80`, finite elements method has a large development, and the analytical solution is almost abandoned. Remarkable results in case of a vertical Rayleigh cantilever beam are in [1] where natural frequencies and mode shapes are studied. Considering shear deformation theory, Thuc [2] has determined the mode shapes with for axial load values up cu 0.5 of critical load. A relevant book with a large approach of vibrating axial loaded structures had been done by Virgin [3].

In this study, we investigate the dynamic response of elastic beam with axial load (geometrically non-linear vibration). Therefore, we take one-span Bernoulli-Euler beam loaded with dynamic loads - distributed forces and concentrate forces.

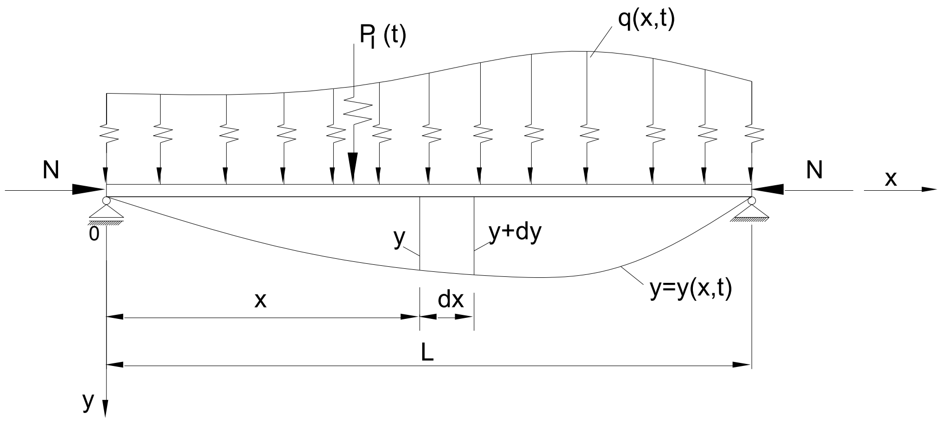


Figure 1. One spam beam loaded with external forces

The axial load (N), flexural rigidity (EI) and the mass density per unit length (ρ) of the beam are constants. For the partial derivate equation, will consider a differential element of beam, considering the external dynamic forces and internal forces [4]:

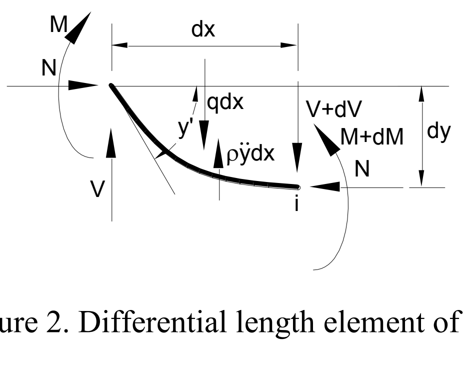


Figure 2. Differential length element of beam

In figure 2, V is the vertical shear force component T. This component, in nonlinear analysis it is tangent at a point x of medium fibre, $y = y(x,t)$.

Thus, we can write

$$\begin{cases} \sum X_i = 0 \\ \sum Y_i = 0 \\ \sum M_i = 0 \end{cases} \Leftrightarrow \begin{cases} N - N = 0 \\ V + dV - V + qdx - \rho\ddot{y}dx = 0 \\ M + dM - M - Vdx - Ndy + qdx\frac{dx}{2} - \rho\ddot{y}dx\frac{dx}{2} = 0 \end{cases} \quad (1)$$

The differential equation that gives the elastic curve for the deflected beam is

$$M = -\frac{EI}{\rho};\ \frac{1}{\rho} \approx y'';\ M = -EI\, y''$$

Further

$$-EI\, y'' = -q + \rho\ddot{y} + Ny''$$

At last, the final expression

$$EI\, y^{IV} + Ny'' + \rho\ddot{y} = q. \quad (2)$$

If external loads are concentrated $P_i(t)$, at sections x_i the Eq. (6) becomes

$$EI\, y^{IV} + Ny'' + \rho\ddot{y} = q + \sum_i P_i\, \delta(x - x_i) \quad (3)$$

where $\delta(x - x_i)$ is the Dirac function.

Comment:

Starting from Eq. (2)

$$\frac{dM}{dx} = T$$

$$T = V + N\, y' \quad (4)$$

and it reveal the influence of axial load (N) in the value of shear force (T) in non-linear calculus.

The partial differential Eq. (7) describes the geometrically non-linear vibration of a beam.

2. Free vibrations. Normal modes shapes of vibrations

The partial derivate equation of free vibrations, due to initial conditions (displacements and velocities) without externals loads is [5]:

$$EI\, y^{IV} + N\, y'' + \rho\, \ddot{y} = 0 \quad (5)$$

For the integration of Eq. (8), we will apply the variables separation technique. Thus, the normal modes shapes of vibrations and the natural frequencies functions should be determined. For that the solution of Eq. (8) will be written:

$$y(x,t) = \phi(x) \cdot \eta(t) \tag{6}$$

So further on the Eq. (8) becomes:

$$EI\, \phi^{IV}(x) \cdot \eta(t) + N\, \phi''(x) \cdot \eta(t) + \rho\, \phi(x) \cdot \ddot{\eta}(t) = 0 \tag{7}$$

divided by $\rho\phi(x) \cdot \eta(t)$:

$$\frac{EI}{\rho} \cdot \frac{\phi^{IV}(x)}{\phi(x)} + \frac{N}{\rho} \cdot \frac{\phi''(x)}{\phi(x)} = -\frac{\ddot{\eta}(t)}{\eta(t)} = \omega^2 \tag{8}$$

In Eq. (11), the two fractions, of different variables, could not be equal only in the case that both fractions are constants. That constant is ω^2 also known as the natural frequency.

Further, from Eq. (8) we can write the following:

$$\ddot{\eta}(t) + \omega^2 \eta(t) = 0 \tag{9}$$

with the solution

$$\eta(t) = A \cdot \sin(\omega t + \varphi) \tag{10}$$

and it concludes that the free dynamic response is harmonic and dependent of the frequency ω.

The first equality of Eq. (11) is written:

$$\phi^{IV}(x) + n^2 \phi''(x) - \alpha^4 \phi(x) = 0 \tag{11}$$

where

$$n^2 = \frac{N}{EI}\,; \alpha^4 = \frac{\omega^2 \rho}{EI} \tag{12}$$

The natural frequencies functions $\omega_j, \phi_j(x)$ equation that results from Eq. (11) is dependent of boundary conditions. The differential equation and the boundary conditions are also known as a Sturm-Liouville problem.

$$\begin{cases} \phi^{IV}(x) + n^2 \phi''(x) - \alpha^4 \phi(x) = 0 \\ \text{" + "} \\ \text{(homogeneous boundary conditions)} \end{cases} \tag{13}$$

Solving this problem will determine the natural frequencies functions, which are non-zeros solutions. Thus, the solution of Eq. (13) it is replace by his own solution.

3. The homogeneous solution of differential equation

Eq. (13) is a linear equation, homogeneous, with constant coefficients, which has the particular solutions by the form of $e^{\lambda x}$. So, we can write:

$$\lambda^4 + n^2\lambda^2 - \alpha^4 = 0 \tag{14}$$

with the solutions

$$\begin{aligned} \lambda_1^2 &= \frac{\sqrt{n^4 + 4\alpha^4} - n^2}{2} > 0 \\ \lambda_2^2 &= -\frac{\sqrt{n^4 + 4\alpha^4} + n^2}{2} < 0 \end{aligned} \tag{15}$$

Thus, the general solution of Eq. (17) is

$$\phi(x) = A \cdot \cosh(\lambda_1 x) + B \cdot \sinh(\lambda_1 x) + C \cdot \cos(\lambda_2 x) + D \cdot \sin(\lambda_2 x) \tag{16}$$

and the constants A, B, C and D will be determined from the boundary conditions of the problem.

4.Examples. The six fundamental cases.

The homogeneous boundary conditions of Sturm-Liouville problem - Eq. (16) – involve the derivatives of general solution – Eq. (16) [6]:

$$\begin{aligned} \phi'(x) &= A\lambda_1 \cdot \sinh(\lambda_1 x) + B\lambda_1 \cdot \cosh(\lambda_1 x) - C\lambda_2 \cdot \sin(\lambda_2 x) + D\lambda_2 \cdot \cos(\lambda_2 x) \\ \phi''(x) &= A\lambda_1^2 \cdot \cosh(\lambda_1 x) + B\lambda_1^2 \cdot \sinh(\lambda_1 x) - C\lambda_2^2 \cdot \cos(\lambda_2 x) - D\lambda_2^2 \cdot \sin(\lambda_2 x) \\ \phi'''(x) &= A\lambda_1^3 \cdot \sinh(\lambda_1 x) + B\lambda_1^3 \cdot \cosh(\lambda_1 x) + C\lambda_2^3 \cdot \sin(\lambda_2 x) - D\lambda_2^3 \cdot \cos(\lambda_2 x) \end{aligned} \tag{17}$$

The six fundamental cases in our study are:

- simply supported beam
- free beam at both ends
- cantilevered beam
- double clamped beam
- clamped – simply supported
- free bearing - free at the other end

4.1 Simply supported beam

According to Eq. (16), the differential equation is replaced by his own solution – Eq. (19). Thus, the Sturm-Liouville problem is:

$$\begin{cases} \phi(x) = A \cdot \cosh(\lambda_1 x) + B \cdot \sinh(\lambda_1 x) + C \cdot \cos(\lambda_2 x) + D \cdot \sin(\lambda_2 x) \\ \phi(0) = 0 \\ \phi''(0) = 0 \\ \phi\,(l) = 0 \\ \phi''(l) = 0 \end{cases} \tag{18}$$

Thus, the system is:

$$\begin{cases} A + C = 0 \\ A\lambda_1^2 - C\lambda_2^2 = 0 \\ A \cdot \cosh(\lambda_1 l) + B \cdot \sinh(\lambda_1 l) + C \cdot \cos(\lambda_2 l) + D \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1^2 \cdot \cosh(\lambda_1 l) + B\lambda_1^2 \cdot \sinh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \end{cases} \quad (19)$$

It is a linear equation system, homogeneous. To avoid the zero solution, we must condition the determinant to be equal to zero. Therefore:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1^2 & 0 & -\lambda_1^2 & 0 \\ \cosh(\lambda_1 l) & \sinh(\lambda_1 l) & \cos(\lambda_2 l) & \sin(\lambda_2 l) \\ \lambda_1^2\cosh(\lambda_1 l) & \lambda_1^2\sinh(\lambda_1 l) & -\lambda_2^2\cosh(\lambda_2 l) & -\lambda_2^2\sin(\lambda_2 l) \end{vmatrix} = 0 \quad (20)$$

Only that, the system Eq. (22) can be reduced to a two-equation system with two variables. The first two equations of Eq. (22) can be separated and there determinant is:

$$\begin{vmatrix} 1 & 1 \\ \lambda_1^2 & -\lambda_2^2 \end{vmatrix} = -(\lambda_1^2 + \lambda_2^2) \neq 0 \quad (21)$$

Thus, $A = C = 0$.

So, further Eq. (22) can be written:

$$\begin{cases} B \cdot \sinh(\lambda_1 l) + D \cdot \sin(\lambda_2 l) = 0 \\ B\lambda_1^2 \cdot \sinh(\lambda_1 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \end{cases} \quad (22)$$

this system of homogeneous equations is with two variables $B \cdot \sinh(\lambda_1 l)$ and $D \cdot \sin(\lambda_2 l)$. The determinant of this equations system is the same, as Eq. (24), not equal to zero, it results:

$$B \cdot \sinh(\lambda_1 l) = 0\,; D \cdot \sin(\lambda_2 l) = 0. \quad (23)$$

Only that,

$$\sinh(\lambda_1 l) \neq 0$$

thus $B = 0$, farther more the $\phi(x) \neq 0$, thus $D \neq 0$. Therefore, we can write:

$$\sin(\lambda_2 l) = 0 \quad (24)$$

The Eq. (27) is also known as the frequency equation and D is an arbitrary constant. The mode shape functions are:

$$\phi_j(x) = \sin(\lambda_2 x) \quad (25)$$

From Eq. (27) results the natural frequencies values:

$$\lambda_2 l = j\pi; j = 1, 2, 3, \ldots \quad (26)$$

$$\lambda_2 = \frac{j\pi}{l};\ \lambda_2^2 = \frac{j^2\pi^2}{l^2} \quad (27)$$

From Eq. (18), with the absolute value of $|\lambda_2^2|$, results:

$$\frac{\sqrt{n^4+4\alpha^4}+n^2}{2}=\frac{j^2\pi^2}{l^2}$$

To obtain natural frequencies values ω_j, also α_j^4, will proceed further:

$$\alpha_j^4=\frac{j^4\pi^4}{l^4}\left(1-\frac{n^2l^2}{j^2\pi^2}\right)$$

As a result Eq. (15) can be written as follows, without any comments

$$\omega_j^2=\frac{EI}{\rho}\left[\frac{j^4\pi^4}{l^4}\left(1-\frac{n^2l^2}{j^2\pi^2}\right)\right] \tag{28}$$

or

$$\omega_j^2=\frac{EI}{\rho}\frac{j^4\pi^4}{l^4}\left(1-\frac{\nu^2}{j^2\pi^2}\right)$$

$$\omega_j=\frac{j^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho}\left(1-\frac{\nu^2}{j^2\pi^2}\right)} \tag{29}$$

where ν is the compression factor.

If the exterior forces have no axial load $(N=0)$, natural frequencies values ω_j are:

$$\omega_j=\frac{j^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho}} \tag{30}$$

Comment

The frequencies of a beam with axial load are smaller than frequencies of a beam without axial load. That occurs in reducing the rigidity of the beam, due the axial load. This observation remain the same for all others cases of boundary conditions.

Due to this reduction of rigidity, the pulsations values decrease to value 0, where the factor of compression has the value of critical load.

The shape functions are showed in the next figure.

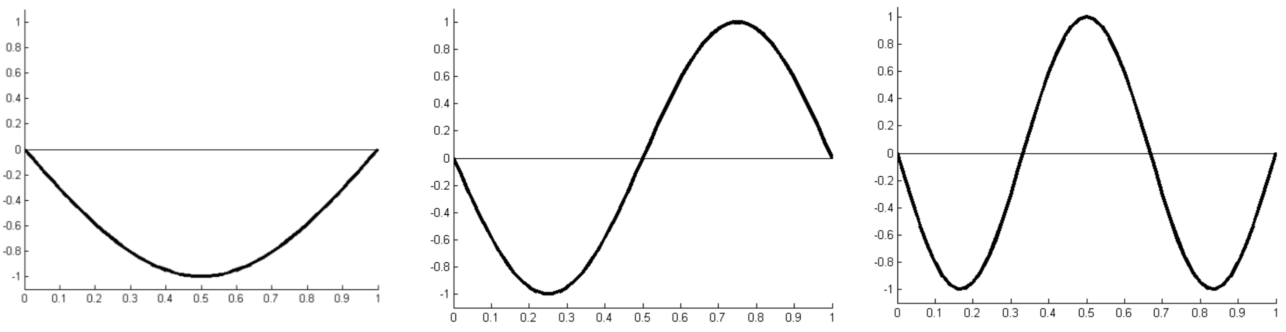


Figure 3. Normal mode shapes of simply supported beam

4.2 Free beam at both ends

Boundary conditions for a free beam at the bought ends, considering that the bending moments and shear forces are zero ($M = 0; T = 0$), are:

$$\begin{cases} \phi''(0) = 0 \\ EI \cdot \phi'''(0) + N \cdot \phi'(0) = 0 \\ \phi''(l) = 0 \\ EI \cdot \phi'''(l) + N \cdot \phi'(l) = 0 \end{cases} \tag{31}$$

Where N is the axial load. As a result, Eq. (20) can be written as follows

$$\begin{cases} A\lambda_1^2 - C\lambda_2^2 = 0 \\ B\lambda_1^3 - D\lambda_2^3 + n^2(B\lambda_1 - D\lambda_2) = 0 \\ A\lambda_1^2 \cdot ch(\lambda_1 l) + B\lambda_1^2 \cdot sh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1^3 \cdot sh(\lambda_1 l) + B\lambda_1^3 \cdot ch(\lambda_1 l) + C\lambda_2^3 \cdot \sin(\lambda_2 l) - D\lambda_2^3 \cdot \cos(\lambda_2 l) + \\ +n^2[A\lambda_1 \cdot sh(\lambda_1 l) + B\lambda_1 \cdot ch(\lambda_1 l) - C\lambda_2 \cdot \sin(\lambda_2 l) + D\lambda_2 \cdot \cos(\lambda_2 l)] = 0 \end{cases} \tag{32}$$

taking account of Eq. (32)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{33}$$

and the coefficients are:

$$\begin{aligned} a_{11} &= \lambda_1^2 \cdot [ch(\lambda_1 l) - \cos(\lambda_2 l)] \\ a_{12} &= \lambda_1 \cdot \left[\lambda_1 sh(\lambda_1 l) - \lambda_2 \cdot \frac{\lambda_1^2 + n^2}{n^2 - \lambda_1^2} \sin(\lambda_2 l)\right] \\ a_{21} &= \lambda_1 \cdot \left[(\lambda_1^2 + n^2) sh(\lambda_1 l) + \frac{\lambda_1}{\lambda_2}(\lambda_1^2 - n^2)\sin(\lambda_2 l)\right] \\ a_{22} &= \lambda_1(\lambda_1^2 + n^2)[ch(\lambda_1 l) - \cos(\lambda_2 l)]. \end{aligned}$$

Solving this determinant, it results the frequencies equation:

$$2\lambda_1^3\lambda_2^3[1 - ch(\lambda_1 l)\cos(\lambda_1 l)] + (\lambda_2^6 - \lambda_1^6) sh(\lambda_1 l)\sin(\lambda_1 l) = 0 \tag{34}$$

Further, with the notation $z = (\lambda_2 l)$; the primary unknown variable and $(\lambda_1 l)^2 = z^2 - \nu^2$

Thus, Eq. (34) becomes

$$\begin{aligned} 2\left(\sqrt{z^2 - \nu^2}\right)^3 \cdot z^3 \cdot \left[1 - ch\left(\sqrt{z^2 - \nu^2}\right)\cos(z)\right] + \\ + \left[z^6 - \left(\sqrt{z^2 - \nu^2}\right)^6\right] sh\left(\sqrt{z^2 - \nu^2}\right)\sin(z) = 0 \end{aligned} \tag{35}$$

This equation is also known as the frequency equation in the z variable. The solutions of these equations are obtained for explicit values of compression factor ν. Thus, it is obtained the solutions:

$$0 < z_1 < z_2 < z_3 < \cdots$$

$$0 < (\lambda_1 l)_1 < (\lambda_1 l)_2 < (\lambda_1 l)_3 < \cdots (\lambda_1 l)_j; j = 1,2,3, \ldots$$

The natural frequencies value, ω_j, can be determined from the above solutions. At this we associate the shape mode function $\phi_j(x)$.

The couple $\left(\omega_j, \phi_j(x)\right)$ defines a normal mode shape of vibration. Thus, to determinate of a normal mode shape, we start from second expression from Eq. (39). In other cases of boundary conditions, the steps are the same.

where the notation was use

$$k_j = \frac{(\lambda_2 l)[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_2 l) - (\lambda_1 l)\sin(\lambda_2 l)} \quad (36)$$

The variables and constants from above have the index number "j", so we can conclude:

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \sinh(\lambda_1 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j^2 \cos(\lambda_1 l)_j \cdot \frac{x}{l} - \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j^3 k_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \quad (37)$$

The shape functions at $\nu = 0$ and $\nu = \nu_{crt}$ for the first three mode shapes are showed in the next figure.

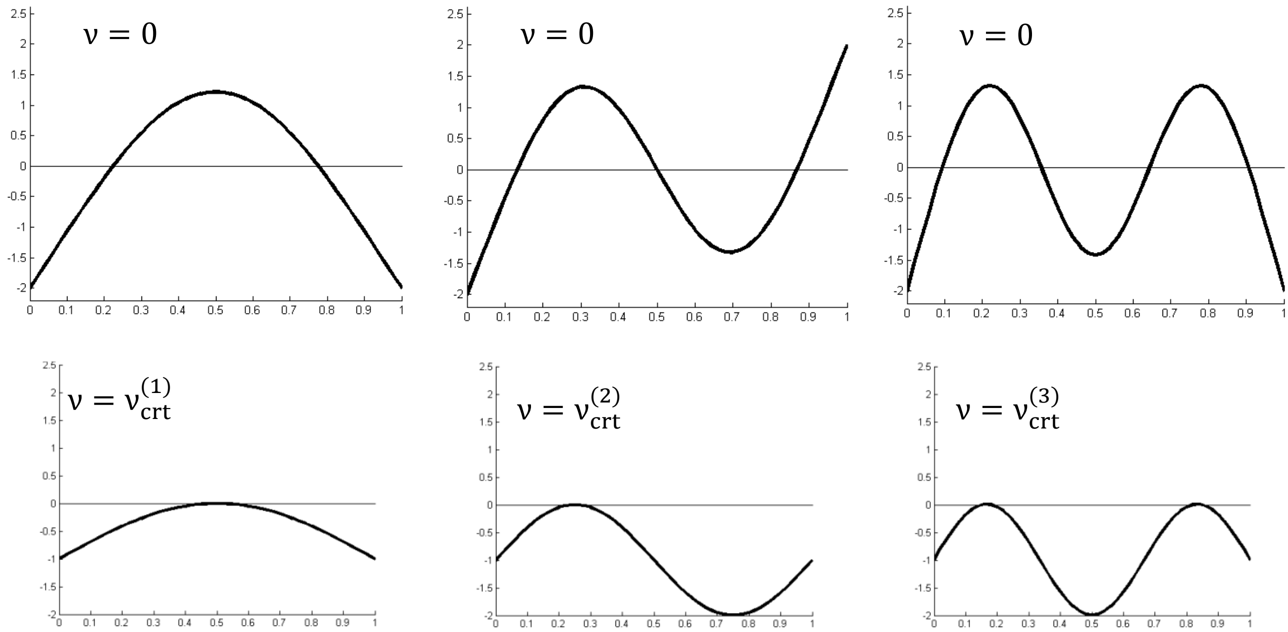


Figure 4. Normal mode shapes of free beam at both ends

4.3 Cantilevered beam

Boundary conditions for a cantilevered beam, with the clamping at the left end, are:

$$\begin{cases} \phi(0) = 0 \\ \phi'(0) = 0 \\ \phi''(l) = 0 \\ EI \cdot \phi'''(l) + N \cdot \phi'(l) = 0 \end{cases} \quad (38)$$

So, from Eq. (19) and Eq. (20)

$$\begin{cases} A + C = 0 \\ B\lambda_1 + D\lambda_2 = 0 \\ A\lambda_1^2 \cdot ch(\lambda_1 l) + B\lambda_1^2 \cdot sh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1(\lambda_1^2 + n^2) \cdot sh(\lambda_1 l) + B\lambda_1(\lambda_1^2 + n^2) \cdot ch(\lambda_1 l) - C\lambda_1(\lambda_1^2 + n^2) \cdot \sin(\lambda_2 l) - \\ \qquad -D\lambda_1(\lambda_1^2 + n^2) \cdot \cos(\lambda_2 l) = 0 \end{cases} \tag{39}$$

It is easy to show, from the first two equations, that

$$\begin{cases} C = -A \\ D\lambda_2 = B\lambda_1 \end{cases}; D = -\frac{\lambda_1}{\lambda_2} B$$

and further

$$\begin{cases} A[\lambda_1^2\, ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l)] + B\lambda_1[\lambda_1\, sh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l)] = 0 \\ A[\lambda_1^3\, sh(\lambda_1 l) - \lambda_2^3 \sin(\lambda_2 l) + n^2\lambda_1 sh(\lambda_1 l) + n^2\lambda_2 \sin(\lambda_2 l)] + \\ \qquad +B\lambda_1[\lambda_1^2 ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l) + n^2 ch(\lambda_1 l) - n^2 \cos(\lambda_2 l)] = 0 \end{cases} \tag{40}$$

For nontrivial solution, we must condition the determinant to be equal to zero. Therefore:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{41}$$

And

$$\begin{aligned} a_{11} &= \lambda_1^2 ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l) \\ a_{12} &= \lambda_1\, sh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l) \\ a_{21} &= \lambda_1^3\, sh(\lambda_1 l) - \lambda_2^3 \sin(\lambda_2 l) + n^2[ch(\lambda_1 l) - \cos(\lambda_2 l)] \\ a_{22} &= \lambda_1^2 ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l) + n^2[ch(\lambda_1 l) - \cos(\lambda_2 l)] \end{aligned}$$

Further more

$$[\lambda_1^2 ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_2 l)]^2 + n^2[\lambda_1^2 ch(\lambda_1 l) + \lambda_2^2 \cos(\lambda_1 l)] \cdot [ch(\lambda_1 l) - \cos(\lambda_2 l)] -$$

$$-[\lambda_1\, sh(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l)] \cdot [\lambda_1^3\, sh(\lambda_1 l) - \lambda_2^3 \sin(\lambda_2 l) + n^2 ch(\lambda_1 l) - n^2 \cos(\lambda_2 l)] = 0$$

$$\lambda_1^4 ch^2(\lambda_1 l) + 2\lambda_1^2\lambda_2^2 ch(\lambda_1 l)\cos(\lambda_2 l) + \lambda_2^4 \cos^2(\lambda_2 l) + n^2\lambda_1^2 ch^2(\lambda_1 l) - n^2\lambda_1^2 ch(\lambda_1 l)\cos(\lambda_2 l) +$$

$$+n^2\lambda_2^2 ch(\lambda_1 l)\cos(\lambda_2 l) - n^2\lambda_2^2 \cos^2(\lambda_2 l) - \lambda_1^4 sh^2(\lambda_1 l) + \lambda_1\lambda_2^3\, sh(\lambda_1 l) \sin(\lambda_2 l) -$$

$$-\lambda_1^3\lambda_2\, sh(\lambda_1 l) \sin(\lambda_2 l) + \lambda_2^4 \sin^2(\lambda_2 l) - n^2\lambda_1^2 sh^2(\lambda_1 l) - 2n^2\lambda_1\lambda_2\, sh(\lambda_1 l) \sin(\lambda_2 l) -$$

$$-n^2\lambda_2^2 \sin^2(\lambda_2 l) = 0$$

The argument of $\lambda_2 x$ from Eq. (19) results from expression $\lambda_2^2 = |\lambda_2^2|$ in the second expression of Eq. (18), thus

$$\lambda_1^2 = \frac{\sqrt{n^4 + 4\alpha^4} - n^2}{2}; \quad \lambda_2^2 = \frac{\sqrt{n^4 + 4\alpha^4} + n^2}{2} \tag{42}$$

That determines

$$\lambda_1^2+\lambda_2^2=\sqrt{n^4+4\alpha^4};\ \lambda_1^2-\lambda_2^2=n^2;\ \ \lambda_1^2\lambda_2^2=\alpha^4;\ \ \lambda_1\lambda_2=\alpha^2$$

$$\lambda_1^4+\lambda_2^4+2\lambda_1^2\lambda_2^2=n^4+4\alpha^4 \tag{43}$$

$$\lambda_1^4+\lambda_2^4=n^4+4\alpha^4-2\alpha^4;\ \ \lambda_1^4+\lambda_2^4=n^4+2\alpha^4$$

we obtain, and multiply it with l^4

$$(\lambda_1 l)^4+(\lambda_2 l)^4+(nl)^2[(\lambda_1 l)^2-(\lambda_2 l)^2]+$$

$$+\left[2(\lambda_1 l)^2(\lambda_2 l)^2+(nl)^2[(\lambda_2 l)^2-(\lambda_1 l)^2]\right]\mathrm{ch}(\lambda_1 l)\cos(\lambda_2 l)+$$

$$+(\lambda_1 l)(\lambda_2 l)[(\lambda_2 l)^2-(\lambda_1 l)^2-2(nl)^2]\,\mathrm{sh}(\lambda_1 l)\sin(\lambda_2 l)=0$$

with the notations:

$$nl=\nu;\quad \lambda_2 l=z;\quad (\lambda_1 l)^2=z^2-n^2$$

the frequencies equation becomes

$$(z^2-\nu^2)^2+z^4-\nu^4+[2(z^2-\nu^2)z^2+\nu^4]\mathrm{ch}\left(\sqrt{z^2-\nu^2}\right)\cos(z)-$$
$$-\nu^2 z\sqrt{z^2-\nu^2}\,\mathrm{sh}\left(\sqrt{z^2-\nu^2}\right)\sin(z)=0 \tag{44}$$

Solutions from Eq. (55)

$$0<z_1<z_2<z_3<\cdots<z_j<\cdots$$
$$0<(\lambda_2 l)_1<(\lambda_2 l)_2<(\lambda_2 l)_3<\cdots<(\lambda_2 l)_j<\cdots$$

from this solution, the natural frequencies ω_j values are determined with Eq. (46)

$$\omega_j=\frac{z_j^2}{l^2}\sqrt{\frac{EI}{\rho}\left(1-\frac{\nu^2}{z_j^2}\right)}$$

The integration constants will be evaluated from only one. They are dependent to each other, so we can assign an arbitrary value to one of them $A=1$. So the normal shapes of vibrations functions can be determined.

$$C=-A$$

$$\frac{B}{A}=-\frac{\lambda_1^2\cosh(\lambda_1 l)+\lambda_2^2\cos(\lambda_2 l)}{\lambda_1^2\sinh(\lambda_1 l)+\lambda_1\lambda_2\sin(\lambda_2 l)}$$

$$B=-\frac{(\lambda_1 l)^2\cosh(\lambda_1 l)+(\lambda_2 l)^2\cos(\lambda_2 l)}{(\lambda_1 l)^2\sinh(\lambda_1 l)+(\lambda_1 l)(\lambda_2 l)\sin(\lambda_2 l)}\cdot A=-k_j A \tag{45}$$

$$D=-\frac{\lambda_1 l}{\lambda_2 l}B=\frac{\lambda_1 l}{\lambda_2 l}k_j A$$

Therefore, the final normal shapes form expression:

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \sinh(\lambda_1 l)_j \cdot \frac{x}{l} - \cos(\lambda_2 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \quad (46)$$

The shape functions at $\nu = 0$ and $\nu = \nu_{crt}$ for the first three mode shapes are showed in the next figure.

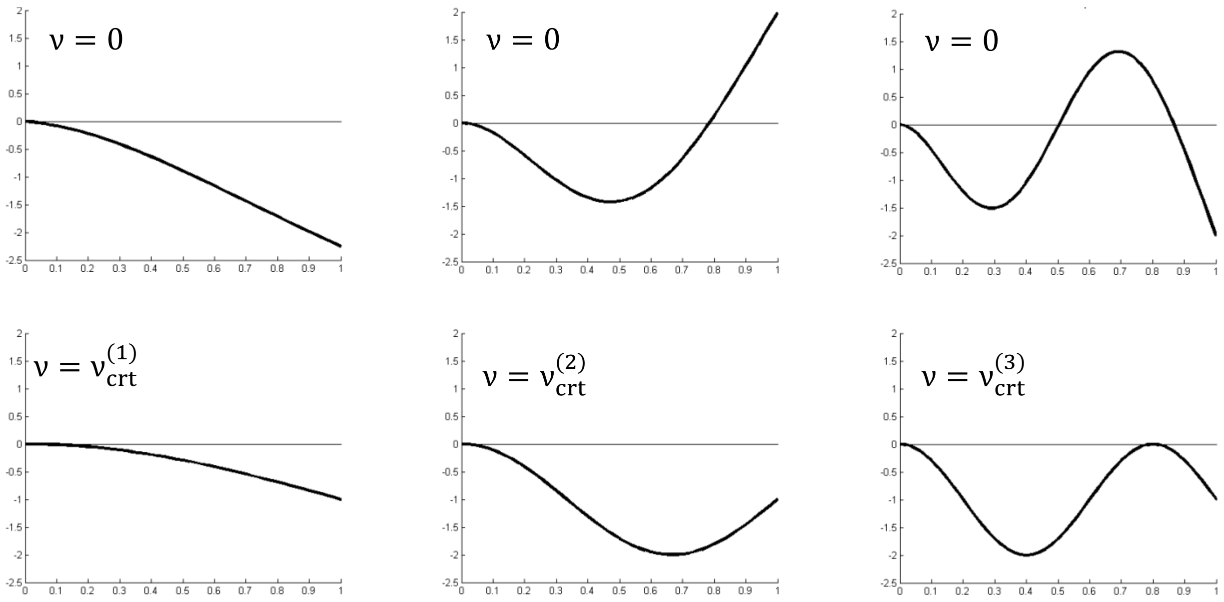


Figure 5. Normal mode shapes of cantilevered beam

4.4 Double clamped beam

Boundary conditions at the ends of beam $x = 0; x = l$ are:

$$\begin{cases} \phi(0) = 0 \\ \phi'(0) = 0 \\ \phi(l) = 0 \\ \phi'(l) = 0 \end{cases} \quad (47)$$

the frequencies equation becomes

$$\left[1 - \cosh\left(\sqrt{z^2 - \nu^2}\right)\cos(z)\right] - \frac{\nu^2}{2z\sqrt{z^2 - \nu^2}} \sinh\left(\sqrt{z^2 - \nu^2}\right)\sin(z) = 0 \quad (48)$$

And with notation

$$k_j = \frac{(\lambda_2 l)[\cosh(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_2 l)\sinh(\lambda_1 l) - (\lambda_1 l)\sin(\lambda_2 l)}, \quad A = 1$$

where the final expression of normal shapes of vibration is

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \cos(\lambda_1 l)_j \cdot \frac{x}{l} - \cos(\lambda_2 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \quad (49)$$

The shape functions at $\nu = 0$ and $\nu = \nu_{crt}$ for the first three mode shapes are showed in the next figure.

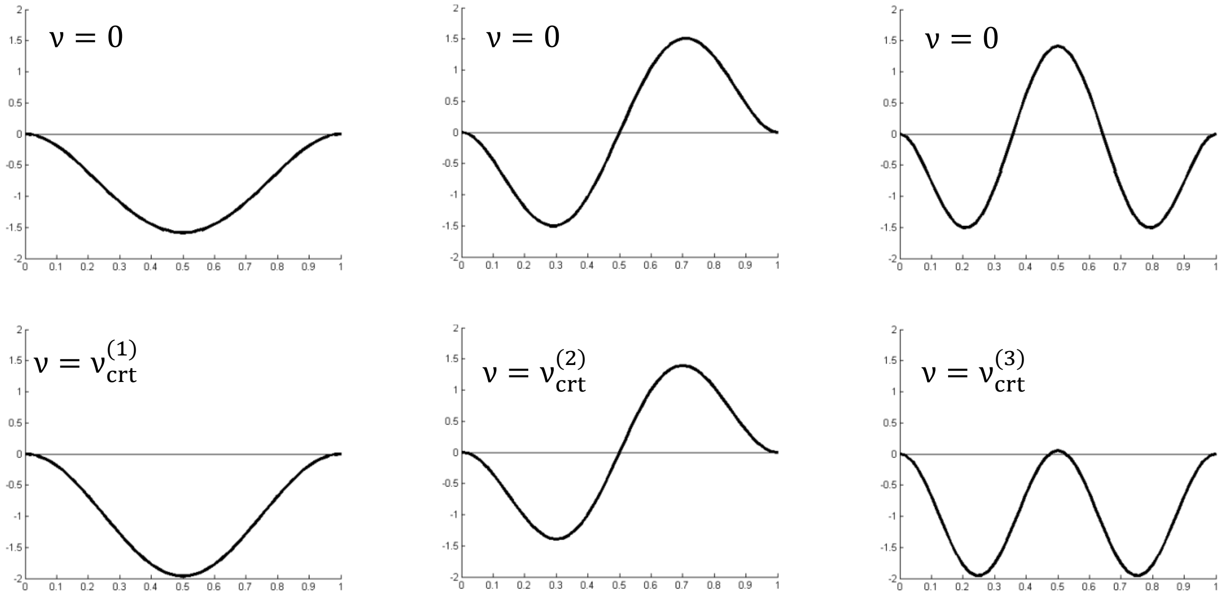


Figure 6. Normal mode shapes of double clamped beam

4.5 Clamped – simply supported beam

In this case, the boundary conditions are:

$$\begin{cases} \phi(0) = 0 \\ \phi'(0) = 0 \\ \phi(l) = 0 \\ \phi''(l) = 0 \end{cases} \tag{50}$$

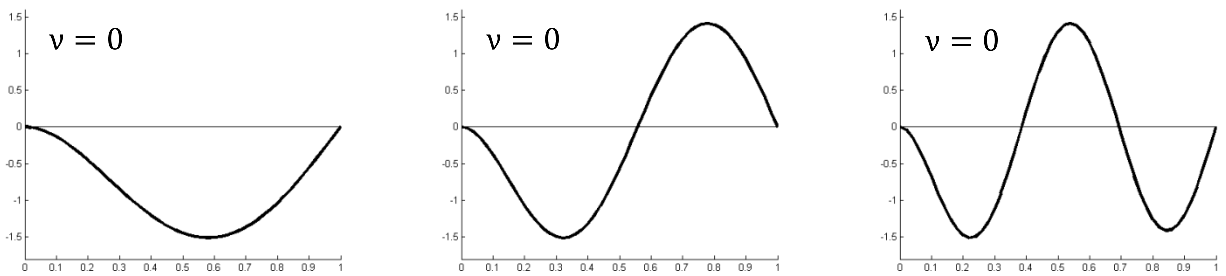
With the frequencies equation:

$$\sqrt{z^2 - \nu^2}\tan(z) - z\tanh(\lambda_1 l) - \nu^2 = 0$$

the natural frequencies ω_j values are determined with Eq. (46) and normal shapes of vibration

$$\phi_j(x) = \cosh(\lambda_1 l)_j \cdot \frac{x}{l} - k_j \cos(\lambda_1 l)_j \cdot \frac{x}{l} - \cos(\lambda_2 l)_j \cdot \frac{x}{l} + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j \sin(\lambda_2 l)_j \cdot \frac{x}{l} \tag{51}$$

The shape functions at $\nu = 0$ and $\nu = \nu_{crt}$ for the first three mode shapes are showed in the next figure.



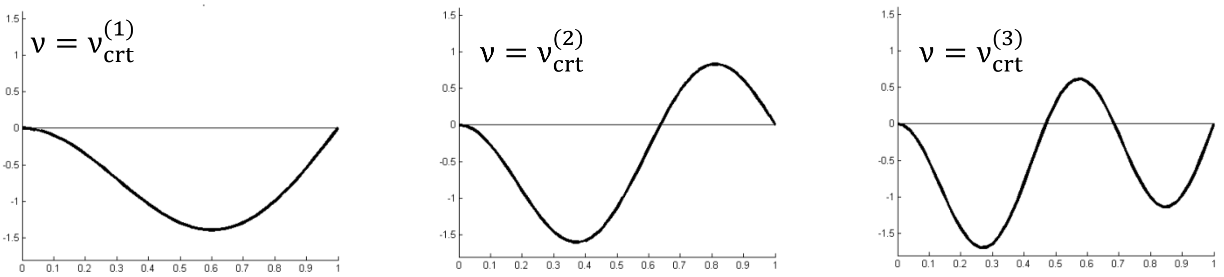


Figure 7. Normal mode shapes of clamped – simply supported beam

4.6 Free bearing- free at the other end

$$\begin{cases} \phi''(0) = 0 \\ EI \cdot \phi'''(0) + N \cdot \phi'(0) = 0 \\ \phi''(l) = 0 \\ EI \cdot \phi'''(l) + N \cdot \phi'(l) = 0 \end{cases} \tag{52}$$

and N is the axial load.

$$\begin{cases} A\lambda_1^2 - C\lambda_2^2 = 0 \\ B\lambda_1^3 - D\lambda_2^3 + n^2(B\lambda_1 - D\lambda_2) = 0 \\ A\lambda_1^2 \cdot ch(\lambda_1 l) + B\lambda_1^2 \cdot sh(\lambda_1 l) - C\lambda_2^2 \cdot \cos(\lambda_2 l) - D\lambda_2^2 \cdot \sin(\lambda_2 l) = 0 \\ A\lambda_1^3 \cdot sh(\lambda_1 l) + B\lambda_1^3 \cdot ch(\lambda_1 l) + C\lambda_2^3 \cdot \sin(\lambda_2 l) - D\lambda_2^3 \cdot \cos(\lambda_2 l) + \\ +n^2[A\lambda_1 \cdot sh(\lambda_1 l) + B\lambda_1 \cdot ch(\lambda_1 l) - C\lambda_2 \cdot \sin(\lambda_2 l) + D\lambda_2 \cdot \cos(\lambda_2 l)] = 0 \end{cases}$$

$$\begin{cases} C = A\dfrac{\lambda_1^2}{\lambda_2^2} \\ D = -B\dfrac{\lambda_1}{\lambda_2} \cdot \dfrac{\lambda_1^2 + n^2}{n^2 - \lambda_1^2} \end{cases}$$

so, the equations system can be rewrite as:

$$\begin{cases} A\lambda_1^2 \cdot [ch(\lambda_1 l) - \cos(\lambda_2 l)] + B\lambda_1 \cdot \left[\lambda_1 sh(\lambda_1 l) - \lambda_2 \cdot \dfrac{\lambda_1^2 + n^2}{n^2 - \lambda_1^2} \sin(\lambda_2 l)\right] = 0 \\ A\lambda_1 \cdot \left[(\lambda_1^2 + n^2) sh(\lambda_1 l) + \dfrac{\lambda_1}{\lambda_2}(\lambda_1^2 - n^2)\sin(\lambda_2 l)\right] + \end{cases} \tag{53}$$

$$+B\lambda_1(\lambda_1^2 + n^2)[ch(\lambda_1 l) - \cos(\lambda_2 l)] = 0$$

the determinant is:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

$$a_{11} = \lambda_1^2 \cdot [\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]$$
$$a_{12} = \lambda_1 \cdot \left[\lambda_1 \mathrm{sh}(\lambda_1 l) - \lambda_2 \cdot \frac{\lambda_1^2 + n^2}{n^2 - \lambda_1^2} \sin(\lambda_2 l)\right]$$
$$a_{21} = \lambda_1 \cdot \left[(\lambda_1^2 + n^2)\mathrm{sh}(\lambda_1 l) + \frac{\lambda_1}{\lambda_2}(\lambda_1^2 - n^2)\sin(\lambda_2 l)\right]$$
$$a_{22} = \lambda_1(\lambda_1^2 + n^2)[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]$$

$$\lambda_1^3\lambda_2^3[\mathrm{ch}^2(\lambda_1 l) - \mathrm{sh}^2(\lambda_1 l)] + \lambda_1^3\lambda_2^3[\cos^2(\lambda_2 l) + \sin^2(\lambda_2 l)] -$$
$$-2\lambda_1^3\lambda_2^3 \mathrm{ch}(\lambda_1 l)\cos(\lambda_2 l) + (\lambda_2^6 - \lambda_1^6)\mathrm{sh}(\lambda_1 l)\sin(\lambda_2 l) = 0$$

$$2\lambda_1^3\lambda_2^3[1 - \mathrm{ch}(\lambda_1 l)\cos(\lambda_1 l)] + (\lambda_2^6 - \lambda_1^6)\mathrm{sh}(\lambda_1 l)\sin(\lambda_1 l) = 0 \quad (54)$$

with the notations $z = (\lambda_2 l)$, and $(\lambda_1 l)^2 = z^2 - \nu^2$

$$2\left(\sqrt{z^2 - \nu^2}\right)^3 \cdot z^3 \cdot \left[1 - \mathrm{ch}\left(\sqrt{z^2 - \nu^2}\right)\cos(z)\right] + \\ + \left[z^6 - \left(\sqrt{z^2 - \nu^2}\right)^6\right] \mathrm{sh}\left(\sqrt{z^2 - \nu^2}\right)\sin(z) = 0 \quad (55)$$

with the solutions

$$0 < z_1 < z_2 < z_3 < \cdots < z_j < \cdots$$
$$0 < (\lambda_2 l)_1 < (\lambda_2 l)_2 < (\lambda_2 l)_3 < \cdots < (\lambda_2 l)_j < \cdots$$
$$\omega_j = \frac{z_j^2}{l^2}\sqrt{\frac{Ei}{\rho}\left(1 - \frac{\nu^2}{z_j^2}\right)}$$

The normal shades functions

$$C = \frac{\lambda_1^2}{\lambda_2^2} A$$

$$D = \frac{\lambda_1}{\lambda_2} \cdot \frac{{\lambda_1}^2 + \nu^2}{{\lambda_2}^2 - \nu^2} \cdot B$$
$$\frac{B}{A} = -\frac{\lambda_1 \mathrm{sh}(\lambda_1 l) + \lambda_2 \sin(\lambda_2 l)}{\lambda_1[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]}$$
$$B = -\frac{(\lambda_1 l)[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_1 l)\mathrm{sh}(\lambda_2 l) - (\lambda_2 l)\frac{(\lambda_1 l)^2 + \nu^2}{(\lambda_2 l)^2 - \nu^2}\sin(\lambda_2 l)} \cdot A = -k_j A$$

$$C = \frac{(\lambda_1 l)^2}{(\lambda_2 l)^2} A$$

$$D = -\frac{\lambda_1 l}{\lambda_2 l} k_j A$$

and the value of k_j is

$$k_j = \frac{(\lambda_2 l)[\mathrm{ch}(\lambda_1 l) - \cos(\lambda_2 l)]}{(\lambda_2 l)\mathrm{sh}(\lambda_1 l) - (\lambda_1 l)\sin(\lambda_2 l)}$$

In the end, the normal shapes functions are

$$\phi_j(x) = \mathrm{ch}\left[(\lambda_1 l)_j \frac{x}{L}\right] - k_j \mathrm{sh}\left[(\lambda_1 l)_j \frac{x}{L}\right] + \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j^2 \cos\left[(\lambda_2 l)_j \frac{x}{l}\right] - \\ - \left(\frac{\lambda_1 l}{\lambda_2 l}\right)_j k_j \sin\left[(\lambda_2 l)_j \frac{x}{l}\right] \quad (56)$$

The shape functions at $\nu = 0$ and $\nu = \nu_{crt}$ for the first three mode shapes are showed in the next figure.

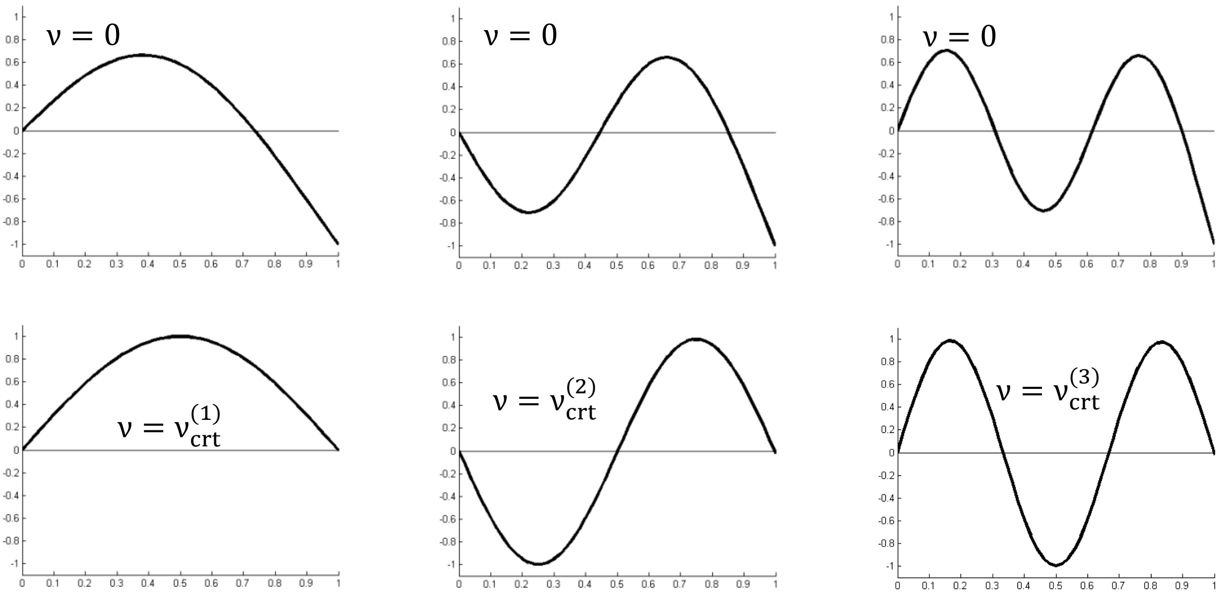


Figure 8. Normal mode shapes of free bearing- free at the other end

1. References

[1] Xi L.Y., Li X.F., Tang G.-J, *Free vibration of standing and hanging gravity-loaded Rayleigh cantilevers.* Elsevier. International Journal of Mechanical Sciences vol. 66, 2013

[2] Thuc P., Huu-Tai Thai *Free vibration of axially loaded rectangular composite beams using refined shear deformation theory*. Elsevier. Composite Structures vol. 94 Issue 11, 2012

[3]Virgin L. *Vibration of axially loaded structures*. New York.Cambridge University Press, 2007

[4]Nowacki W. *Dinamica sistemelor elastice*. Bucureşti: Editura tehnică, 1969.

[5]Bârsan G.M. *Dinamica şi stabilitatea construcţiilor*. Bucureşti: Editura didactică şi pedagogică, 1979.

[6]Ifrim M. *Dinamica structurilor şi inginerie seismică*. Bucureşti: Editura didactică şi pedagogică, 1984.

[7]Ionescu C. *Stabilitatea şi dinamica construcţiilor*. Iaşi: Editura Societăţii Academice Matei-Teiu Botez, 2007