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A Few Data on the Calculation of Beam-Column Joints of Moment Resisting Frames by the Methods of "Theory of Elasticity" (I)

Liviu H. CUCU*[1], Ironim MARŢIAN[2], Daniel I. SUCIU[3]

[1,2,3]*Technical University of Cluj-Napoca, Faculty of Civil Engineering.*
28 Memorandumului Str., 400114, Cluj-Napoca, Romania

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Abstract

In this paper, the authors aim to analyze the state of stress in the beam-column joints of moment resisting frames by accepting two cases: joints as intersection of two bars (frame corners), beam and column and joints as intersection of three bars, two collinear columns and a beam, the latter case being possible to be extended to the joints created as intersection of a column and two collinear beams. The first case has been analyzed either in the "Strength of materials", or in the "Theory of elasticity" as a plane problem in polar coordinates. In the second case, two variants have been taken into consideration: the beam transmits only bending moment and the beam transmits both bending moment and shear force. In both variants, the state of plane stress that results in the beam-column joint of the moment resisting frame has been analytically determined based on the stress formulation, depending on the λ ratio of the height of the column cross section to that of the beam. In this way, the state of stress provided in the works [2] and [5] has been corrected.

Rezumat

În prezenta lucrare, autorii îşi propun să analizeze starea de tensiune în nodurile de cadre acceptând două cazuri: noduri ca intersecţie a două bare (colţuri de cadre), montant (stâlp) şi riglă, şi noduri ca intersecţie a trei bare, doi montanţi în prelungire şi riglă, acest din urmă caz fiind posibil de extins şi la nodurile ca intersecţie a unui montant şi două rigle în prelungire. Primul caz este tratat fie în „Rezistenţa materialelor", fie în „Teoria elasticităţii", ca problemă plană în coordonate polare. În al doilea caz, se consideră două variante: rigla transmite numai moment încovoietor şi rigla transmite atât moment încovoietor, cât şi forţă tăietoare. În ambele variante, starea de tensiune plană care se naşte în nodul de cadru se determină analitic pe baza formulării în tensiuni, în funcţie de raportul λ dintre înălţimea secţiunii montantului şi cea a riglei. În acest fel, se corectează stările de tensiune date în lucrările [2] şi [5].

Keywords: beam-column joint, joint calculation, stress function, state of plane stress

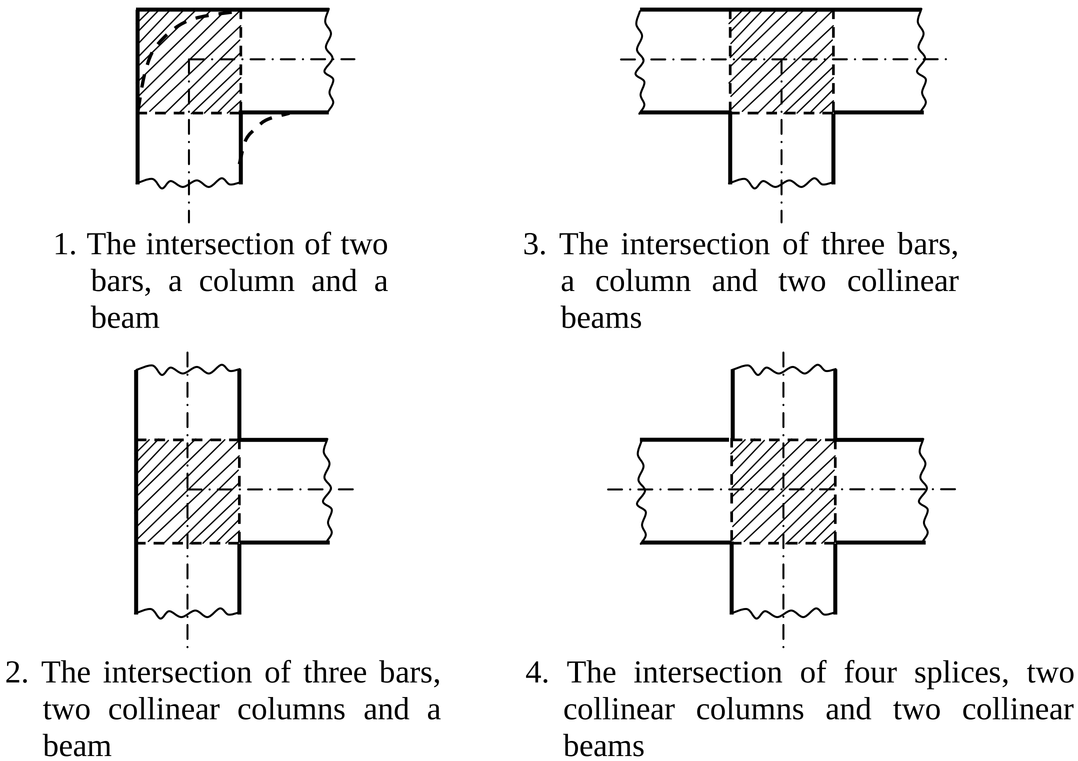
* Corresponding author: Tel./ Fax.: 0722-878769
E-mail address: Hortensiu.Liviu.Cucu@mecon.utcluj.ro

1. Overview

In the process of calculating and performing the connections between the beams and the columns of multi-storey reinforced concrete or steel moment resisting frames, but also in the case of short cantilevers, a special attention must be paid to the joints. In terms of calculations, the state of stress of these joints is complex and will be determined by the methods of "Theory of elasticity", based on the accepted fundamental hypotheses, then the way in which the stresses are dispersed (principal stress trajectories) from the beams to the columns (and vice-versa) is established, in the idea of a proper member reinforcement. Usually, the joints are rigid points, which ensure the kinematic stability, especially in the case of the frames, consisting only of beams and columns. From the practical point of view, in order to obtain a proper beam-column connection, a part of the beam's reinforcement must be inserted in the column and a part of the column's reinforcement must be inserted into the beam. Thus, agglomerations of reinforcing bars can be produced in the beam-column joints, which may lead to inappropriate concreting. Generally, a good detailing of the beam-column joint consists of ensuring appropriate anchorage lengths, avoiding stress concentrations by using haunched connections and easy concreting of these joints [1].

The determination of internal forces on the adjacent bars, which become actions on the beam-column joints, has been done by the "Structural Analysis" methods, depending on the type of the structure (determinate or indeterminate), mainly by imposing compatibility conditions of deformations, eliminating the rigid body displacements in the first case and obtaining the unknown reactions in the second case.

The following types of beam-column joints have been analyzed, in all cases plane stress states ($\delta = 1$) have been occurred, Fig. 1.



1. The intersection of two bars, a column and a beam
2. The intersection of three bars, two collinear columns and a beam
3. The intersection of three bars, a column and two collinear beams
4. The intersection of four splices, two collinear columns and two collinear beams

Fig. 1. Beam-column joints.

2. The simple beam-column joint of moment resisting frames

In the first case (1), the state of stress can be determined, for example, from M_0 (pure bending), considered positively (tends to increase the radius of curvature), by the methods of "Strength of materials", accepting the validity of Bernoulli hypothesis, assuming instead of the joint a circular curved bar with large curvature (small radius of curvature), using the relationship

$$\sigma = \frac{M_0}{S_z} \frac{y}{R_o - y} \quad , \tag{1}$$

where σ is the normal stress at any point on the cross-sectional area situated at a distance y from the neutral axis, R_o is the neutral fiber's radius of curvature, S_z is the first moment of area about the neutral axis; based on the "Theory of elasticity", in polar coordinates, the state of stress that is independent from the angle and the material continuously undistributed around the pole, with the help of the relationships

$$\begin{cases} \sigma_r = -\dfrac{4M_0}{t}\left(\ln\dfrac{b}{a} \cdot \dfrac{a^2b^2}{r^2} + b^2 \ln\dfrac{r}{b} + a^2 \ln\dfrac{a}{r} \right); \\ \sigma_\theta = -\dfrac{4M_0}{t}\left(-\ln\dfrac{b}{a} \cdot \dfrac{a^2b^2}{r^2} + b^2 \ln\dfrac{r}{b} + a^2 \ln\dfrac{a}{r} + b^2 - a^2 \right), \end{cases} \tag{2}$$

where σ_r is the radial stress (tensile stress), σ_θ is the hoop stress on the cross section of the circular curve bar; a is the radius of the internal side fiber and b is the radius of the external side fiber, with the following notation:

$$t = \left(b^2 - a^2\right)^2 - 4a^2b^2\left(\ln\frac{b}{a} \right)^2 .$$

For the particular case $b = 2a$, the extreme normal stresses given by relationships (1) and (2) are acceptable within $\pm 5\%$ limits. The diagrams of σ_r and σ_θ are shown in Fig. 2 [3], [4].

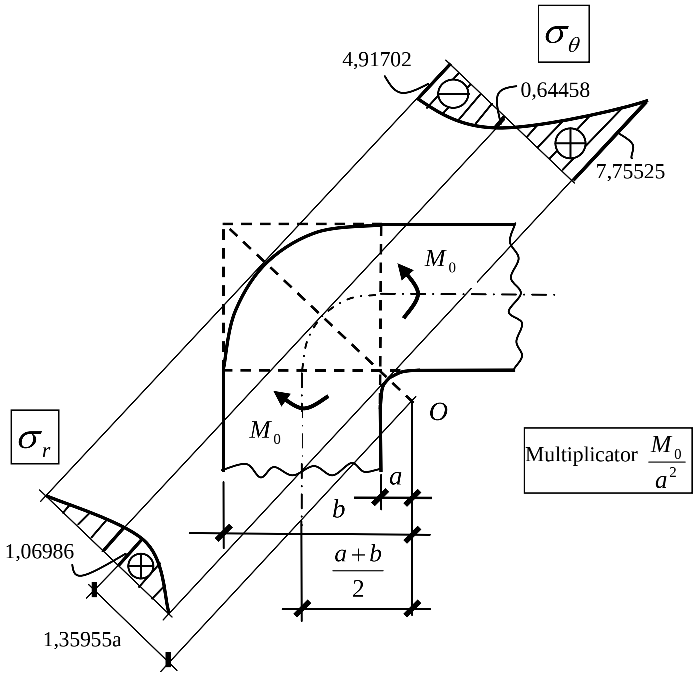


Fig. 2. The diagrams of σ_r and σ_θ for the case $b = 2a$.

3. Two variants for the intersection of three bars in the beam-columns joints

In the second case (2), two variants are considered: I. the beam transmits only bending moment $M_0 = P \cdot 2b$, which determines in the column the shear forces P (Fig. 3,a) ; II. the beam transmits both bending moment M_0 and shear force P, which leads to the tensile axial force P in the column; thus, the bending moment in the beam will be $M_0 = Pa$, while the internal forces in the column will be $T = \lambda^3 P$ and $M = \lambda^2 Pa$, where $\lambda = \frac{a}{b}$, a and b being the dimensions of the joint considered as a rectangular diaphragm (Fig. 3,b).

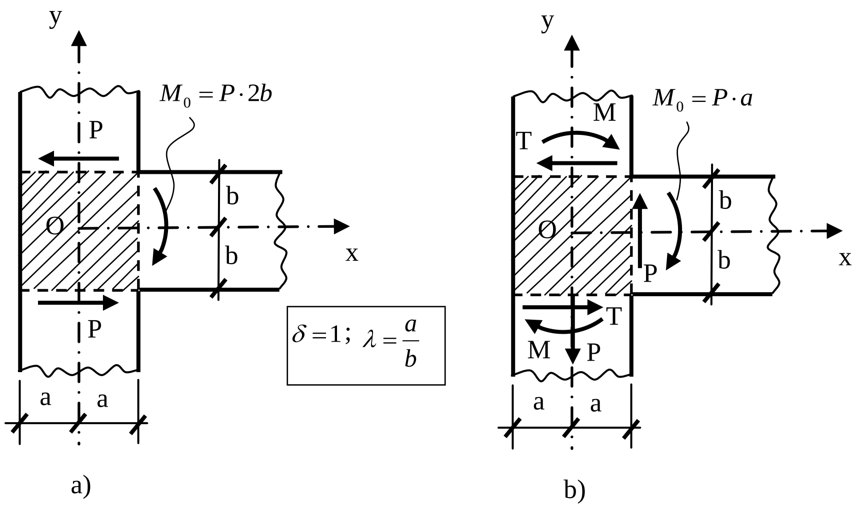


Fig. 3. Two variants for the intersection of three bars in the beam-columns joints.

In the first variant (I), the state of stress in the beam-column joint is determined using the biharmonic stress function (Airy) under the form [2], [5]

$$F(x,y) = P\frac{1}{40a^3b^3}\left[\left(a^2b - 5b^3\right)\left(3a^2 - x^2\right)xy + 5a^2b\left(2ay^2 - x^3 + 3xy^2\right)y + 3b\left(x^2 - \frac{5}{3}y^2\right)x^3y\right], \quad (3)$$

in normalized coordinates $\xi = \frac{x}{a}$, $\eta = \frac{y}{b}$ resulting, in the absence of the body forces

$$\begin{cases} \sigma_x = \dfrac{\partial^2 F}{\partial y^2} = \dfrac{3}{4}\dfrac{P}{a}\lambda(2-\xi)\left(1+\xi^2\right)\eta; \\ \sigma_y = \dfrac{\partial^2 F}{\partial x^2} = \dfrac{3}{4}\dfrac{P}{\lambda a}\left[2\lambda^2\left(\xi^2 - \dfrac{3}{5}\right) + \left(1-\eta^2\right)\right]\xi\eta; \\ \tau_{xy} = -\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{3}{8}\dfrac{P}{a}\left[\lambda^2\left(\xi^2 - \dfrac{1}{5}\right) + 1 - 3\eta^2\right]\left(1-\xi^2\right). \end{cases} \quad (4)$$

The aforementioned relationships for the computation of stresses remain valid only if M_0 and P, acting on the contour, are replaced by mechanically/statically equivalent system of forces, continuously distributed, in accordance with the boundary conditions (Barré de Saint Venant's Principle); thereby σ_x determines a linear diagram for the normal loading (Navier), $\sigma_{x_{\frac{max}{min}}} = \pm\frac{M_0}{W} = \pm 3\frac{P}{a}\lambda$, while $\tau_{xy} = \tau_{yx}$ determines a parabolic diagram (Jurawski) for the tangential loading, $\tau_{yx_{\max}} = \dfrac{3}{2} \cdot \dfrac{P}{1 \cdot 2a} = \dfrac{3P}{4a}$, but since the result of the third relationship (4) is $\tau_{yx_{\max}} = \dfrac{3}{40}\dfrac{P}{a}\left(10+\lambda^2\right)$, it means that the maximum values are approximately equal only if λ takes small values $\lambda < 0{,}5$ (Fig. 4). On the $\eta = \pm 1$ boundary, the resultant of the normal stresses σ_y is:

$$\int_{-1}^{1} \sigma_{y}\Big|_{\eta=\pm 1} \cdot a \cdot d\xi = \pm \frac{3}{2} P\lambda \int_{-1}^{1} \xi \left(\xi^2 - \frac{3}{5} \right) d\xi = 0 \quad . \tag{5}$$

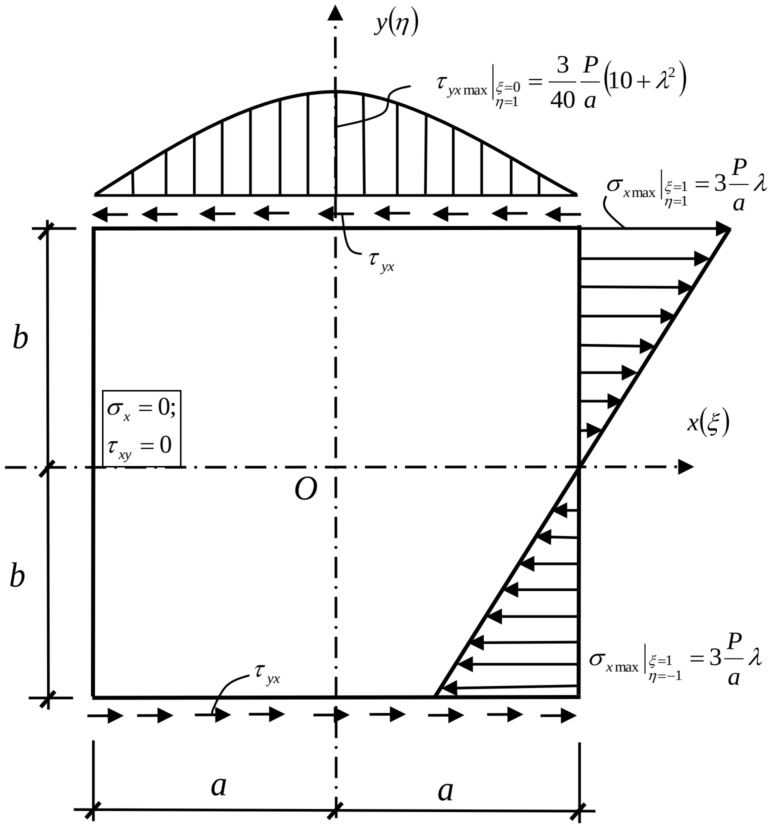


Fig. 4. The state of stress on the joint's contour-Variant I.

Depending on the values of the λ ratio between the dimensions of the middle plane of a beam subjected to a plane state of stress, there are three cases: ***1.*** $\lambda > 2$, that of a straight ordinary beam; ***2.*** $0{,}5 < \lambda < 2$, that of a deep beam; ***3.*** $\lambda < 0{,}5$, that of a short beam (short cantilevers), where the influence of the shear force is considerable in comparison to the bending moment influence. In the latter case, Barré de Saint Venant's Principle can no longer be applied, because by substituting mechanically/statically equivalent systems of forces with others, in order to satisfy the boundary conditions, their effects are not identified anymore. In this last case, the principle of Barré de Saint Venant can no longer apply, since, by replacing certain force systems, equivalent from the mechanical/static point of view, with others, for satisfying the boundary conditions, their effects are not identified anymore. However, such a classification is not categorical, by more rigorous calculations provided by the "Theory of elasticity", a delimitation much closer to reality being possible.

Therefore, if, for instance, $\lambda = 0{,}5$, then $\tau_{yx_{\max}} = \dfrac{3}{40} \cdot \dfrac{P}{a}\left(10 + 0{,}5^2\right) = 0{,}76875\dfrac{P}{a} \cong \dfrac{3}{4} \cdot \dfrac{P}{a} = 0{,}75\dfrac{P}{a}$.

In the second variant (II), the biharmonic stress function is given by [2]

$$F(x,y) = \frac{P}{16ab^3}\left\{\left[y^2 - (a+x)^2 + 8a^2 - 3b^2\right](a+x)^2 y + 2b^3x^2\right\}, \tag{6}$$

the state of stress, keeping the same notations, being

$$\begin{cases} \sigma_x = \dfrac{\partial^2 F}{\partial y^2} = \dfrac{3}{8}\dfrac{P}{a}\lambda^2(1+\xi)^2\eta; \\ \sigma_y = \dfrac{\partial^2 F}{\partial x^2} = \dfrac{P}{8a}\left\{\left[8\lambda^2 - 3 - 6\lambda^2(1+\xi)^2\right]\eta + \eta^3 + 2\right\}; \\ \tau_{xy} = -\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{P\lambda}{8a}(1+\xi)\left[2\lambda^2(1+\xi)^2 + 3(1-\eta^2) - 8\lambda^2\right]. \end{cases} \tag{7}$$

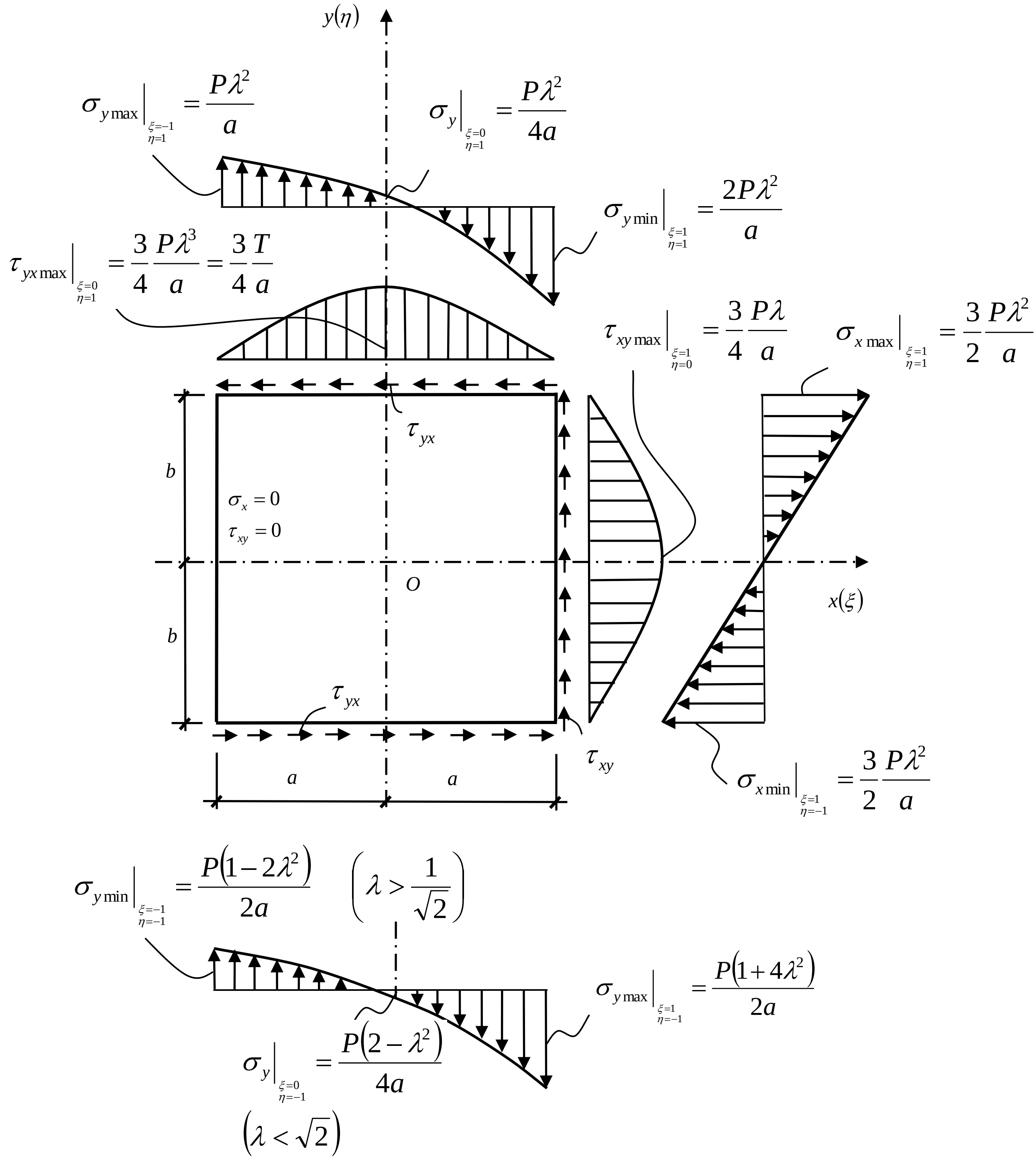


Fig. 5. The state of stress on the joint's contour-Variant II.

From the condition $\tau_{yx_{\max}} = \frac{3}{4}\cdot\frac{P}{a}\lambda^3 = \frac{3}{2}\cdot\frac{T}{1\cdot 2a} = \frac{3T}{4a}$, it actually results that $T = \lambda^3 P$, and from the equation of moment equilibrium about point O, $2M + Pa - T \cdot 2b - Pa = 0$, $M = T \cdot b = \lambda^2 Pa$ is deducted, the force projection equations of the joint being identically satisfied.

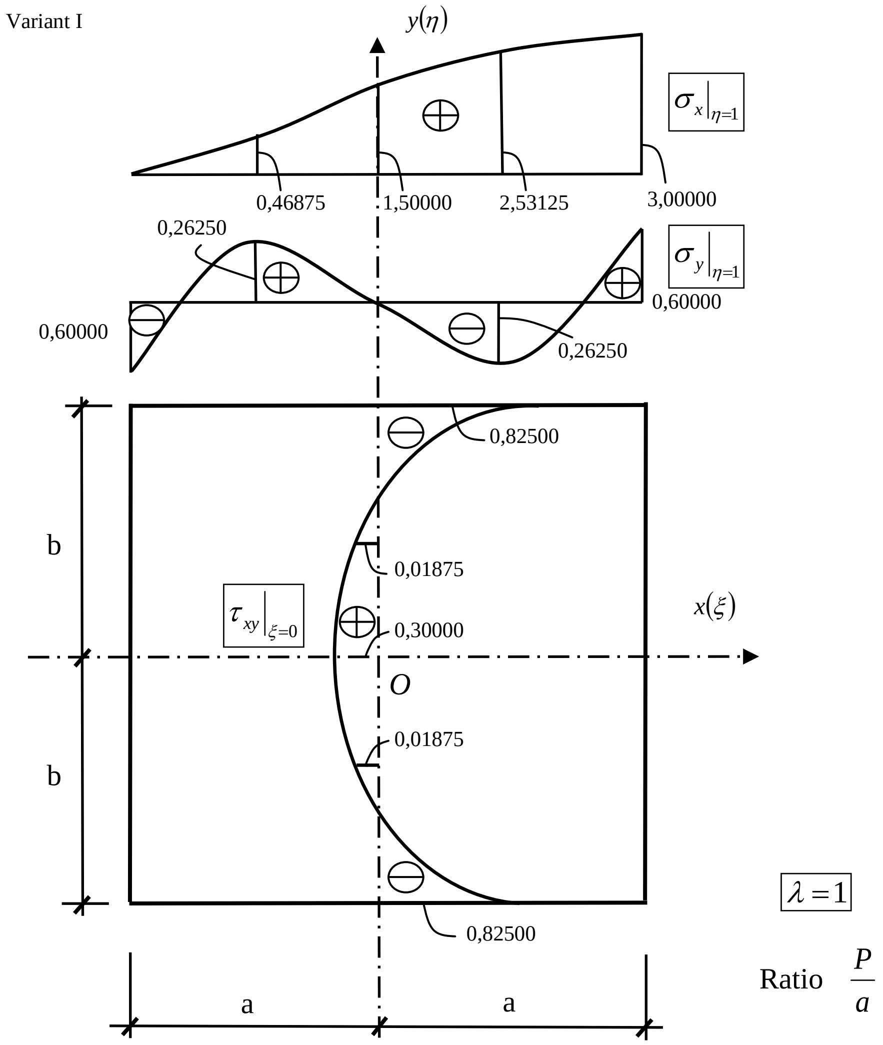


Fig. 6. Characteristic diagrams-Variant I

The state of stress given by the relationships (7) is the real one, if, on the joint's contour, the equivalent continuously distributed forces are introduced, based on satisfying the boundary conditions (Fig. 5). For instance, on the $\xi = +1$ outline

$$\int_{-1}^{1} \tau_{xy}\big|_{\xi=+1} \cdot b \cdot d\eta = \frac{3}{4} P \int_{-1}^{1} \left(1 - \eta^2\right) d\eta = P \tag{8.a}$$

and on the $\eta = \pm 1$ contour

$$\int_{-1}^{1} \sigma_y\big|_{\eta=+1} \cdot a \cdot d\xi = P\lambda^2 \int_{-1}^{1} \left[1 - \frac{3}{4}(1+\xi)^2\right] d\xi = 0, \tag{8.b}$$

respectively

$$\int_{-1}^{1} \sigma_y\big|_{\eta=-1} \cdot a \cdot d\xi = \frac{P}{4} \int_{-1}^{1} \left[2 - 4\lambda^2 + 3\lambda^2(1+\xi)^2\right] d\xi = P, \tag{8.c}$$

namely the axial tensile force;

$$\int_{-1}^{1} \sigma_{y}\big|_{\eta=-1} \cdot a \cdot \xi \cdot a \cdot d\xi = \frac{Pa}{4}\int_{-1}^{1}\left[2-4\lambda^2+3\lambda^2(1+\xi)^2\right]\xi \cdot d\xi = \lambda^2 Pa = M \ , \qquad (8.d)$$

$$\int_{-1}^{1} \tau_{yx}\big|_{\eta=\pm1} \cdot a \cdot d\xi = \frac{P\lambda^3}{4}\int_{-1}^{1}(1+\xi)\left[(1+\xi)^2-4\right]d\xi = -\lambda^3 P = -T \ , \qquad (8.e)$$

hence, the shear force. Several characteristic diagrams for the two variants, are shown in Fig. 6 and 7 and further details are given in Appendix I and II respectively.

Variant II

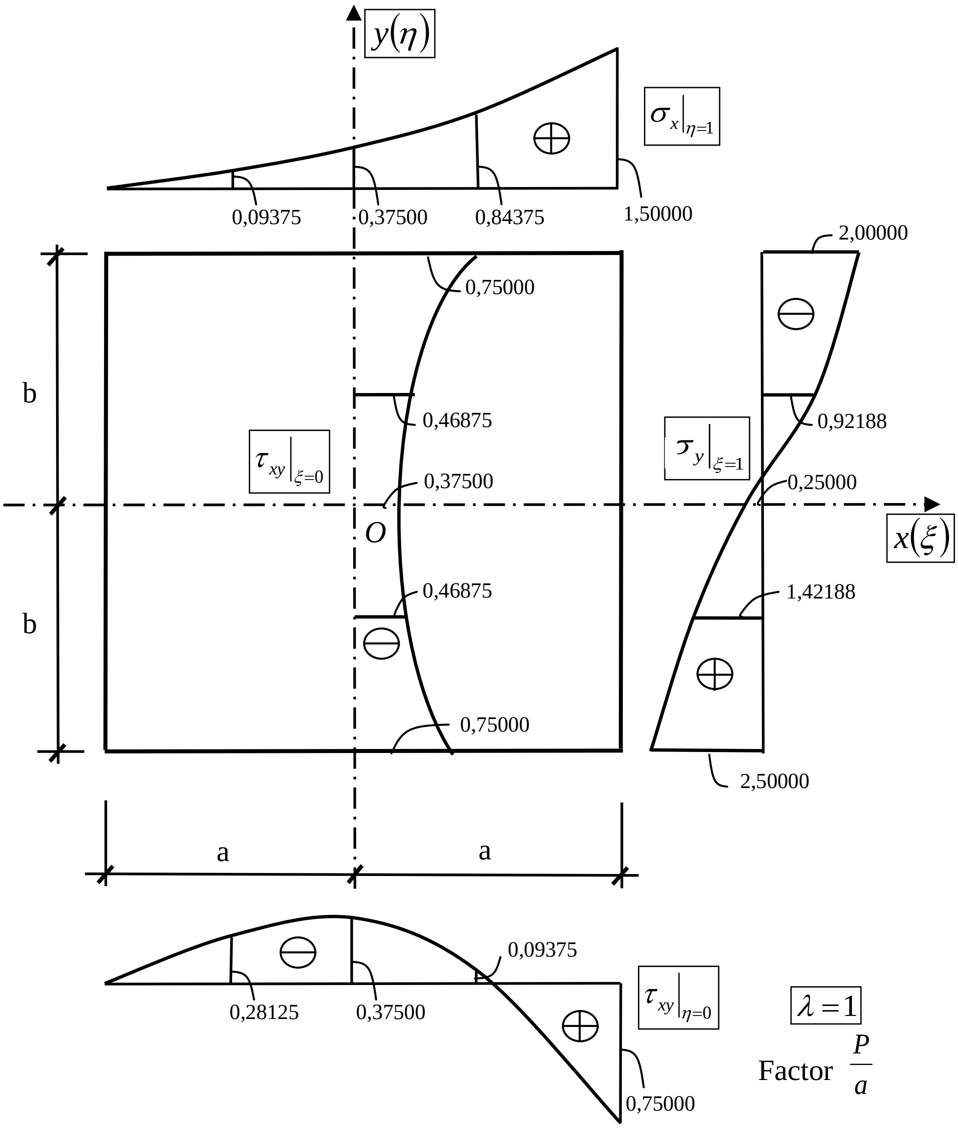


Fig. 7. Characteristic diagrams-Variant II

4. Conclusions

In this way, the polynomial expressions of the biharmonic function $F(x,y)$ $(\nabla^4 F = 0)$ given in [2] and [5] are corrected, also the correct loading of the joint in the second variant is introduced; the states of stress are properly determined in Tables 1 and 2 depicted in the Appendix I and II respectively.

In the first variant (I), the extreme σ_x $(\xi = +1)$ increases with λ from the value of $\pm 0{,}75\frac{P}{a}$ to the value of $\pm 15\frac{P}{a}$, depending on $y(\eta)$, presenting a linear variation (Navier); the σ_y stresses are self-balanced on the $\eta = \pm 1$ contour. The shear stresses $\tau_{xy} = \tau_{yx}$, on the $x(\xi)$ direction, have a parabolic variation (Jurawski), at $\xi = 0$ and $\eta = \pm 1$, τ_{yx} staying constant for the short cantilever ($\lambda \le 0{,}5$), with the value $\tau_{yx_{\max}} = \frac{3}{4} \cdot \frac{P}{a}$, for the deep beams ($0{,}5 < \lambda < 2$) increasing, insignificantly (from $0{,}79219\frac{P}{a}$ to $1{,}05000\frac{P}{a}$), followed, in the case of ordinary beams ($\lambda > 2$), by a very fast increase; τ_{xy}, on the $y(\eta)$ direction, has different signs and at $\xi = \eta = 0$ stays approximately constant ($0{,}37031\frac{P}{a} \div 0{,}30000\frac{P}{a}$) if $\lambda \le 1$, followed by a fast decrease, passing in the range of negative values ($\lambda \ge 3$).

In the second variant (II), extreme σ_x $(\xi = +1)$ increases with λ^2 from the value of $\pm 0{,}09375\frac{P}{a}$ to the value of $\pm 37{,}50000\frac{P}{a}$, depending on η, presenting a linear variation (Navier). The σ_y stresses on the $\eta = +1$ contour lead to a null resultant and to a resultant bending moment equal to $M = \lambda^2 Pa$, while on the $\eta = -1$ contour are specific for the axial force and bending action, but parabolically varying in ξ. The shear stresses $\tau_{xy} = \tau_{yx}$ have Jurawski's type variation laws on the joint's contour, while along along the $Ox(\xi)$ and $Oy(\eta)$ axes, parabolic laws, changing the sign along $Ox(\xi)$ and keeping the sign along $Oy(\eta)$, in this case for $\lambda > 2$ tending to uniformity (b is very small).

For joint types (3) and (4) (Fig. 1), the states of stress can be obtained by properly rotating the axis system and, respectively, by using the superposition of effects for (2) and (3). Experimentally, photoelasticity gives us the possibility to determine the magnitude of the principal normal stresses, their trajectories, even the stress concentrations.

5. References

[1] Mihul A. *Construcţii din beton armat.* Editura Didactică şi Pedagogică, Bucureşti, 1969, pag. 112; 290.

[2] Teodorescu P. P. *Probleme plane în Teoria elasticităţii.* Editura Academiei RPR, Bucureşti, 1961, pag. 386-392.

[3] Marţian I. *Teoria elasticităţii şi plasticităţii pentru constructori.* Editura Universităţii Tehnice din Cluj-Napoca, Cluj-Napoca, 1999, pag. 67-71, 181-185.

[4] Mazilu P., Ţopa N., Ieremia M. *Aplicarea Teoriei elasticităţii şi a plăcilor în calculul construcţiilor.* Editura Tehnică, Bucureşti, 1986, pag. 83-88.

[5] (Coord.) Caracostea A. D. *Manual pentru calculul construcţiilor, Vol. I, Secţ. III - Rezistenţa materialelor şi Teoria elasticităţii.* Editura Tehnică, Bucureşti, 1977, pag. 439-440.

[6] Paillé J-M. *Calcul des structures en béton. Guide d`application.* Afnor et Groupe Eyrolles, Saint-Denis Paris, 2013, pag. 663-691.

Appendix I

Variant I — Ratio P/a — Table 1

$\lambda=1/4$				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	0,75000	0,15000	0,00000
	0,5	0,37500	1,20000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-0,37500	-1,20000	0,00000
	-1	-0,75000	-0,15000	0,00000
0,5	1	0,63281	-0,06600	-0,56200
	0,5	0,31641	0,53000	0,07100
	0	0,00000	0,00000	0,28200
	-0,5	-0,31641	-0,53000	0,07100
	-1	-0,63281	0,06600	-0,56200
0	1	0,37500	0,00000	-0,75469
	0,5	0,18750	0,00000	0,08906
	0	0,00000	0,00000	0,37031
	-0,5	-0,18750	0,00000	0,08906
	-1	-0,37500	0,00000	-0,75469
-0,5	1	0,11719	0,06562	-0,56162
	0,5	0,05859	-0,52969	0,07119
	0	0,00000	0,00000	0,28213
	-0,5	-0,05859	0,52969	0,07119
	-1	-0,11719	-0,06562	-0,56162
-1	1	0,00000	-0,15000	0,00000
	0,5	0,00000	-1,20000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	1,20000	0,00000
	-1	0,00000	0,15000	0,00000

$\lambda=1/2$				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	1,50000	0,30000	0,00000
	0,5	0,75000	0,71250	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-0,75000	-0,71250	0,00000
	-1	-1,50000	-0,30000	0,00000
0,5	1	1,26563	-0,13125	-0,55898
	0,5	0,63281	0,21563	0,07383
	0	0,00000	0,00000	0,28477
	-0,5	-0,63281	-0,21563	0,07383
	-1	-1,26563	0,13125	-0,55898
0	1	0,75000	0,00000	-0,76875
	0,5	0,37500	0,00000	0,07500
	0	0,00000	0,00000	0,35625
	-0,5	-0,37500	0,00000	0,07500
	-1	-0,75000	0,00000	-0,76875
-0,5	1	0,23438	0,13125	-0,55898
	0,5	0,11719	-0,21563	0,07383
	0	0,00000	0,00000	0,28477
	-0,5	-0,11719	0,21563	0,07383
	-1	-0,23438	-0,13125	-0,55898
-1	1	0,00000	-0,30000	0,00000
	0,5	0,00000	-0,71250	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,71250	0,00000
	-1	0,00000	0,30000	0,00000

$\lambda=3/4$				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	2,25000	0,45000	0,00000
	0,5	1,12500	0,60000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-1,12500	-0,60000	0,00000
	-1	-2,25000	-0,45000	0,00000
0,5	1	1,89844	-0,19688	-0,55459
	0,5	0,94922	0,08906	0,07822
	0	0,00000	0,00000	0,28916
	-0,5	-0,94922	-0,08906	0,07822
	-1	-1,89844	0,19688	-0,55459
0	1	1,12500	0,00000	-0,79219
	0,5	0,56250	0,00000	0,05156
	0	0,00000	0,00000	0,33281
	-0,5	-0,56250	0,00000	0,05156
	-1	-1,12500	0,00000	-0,79219
-0,5	1	0,35156	0,19688	-0,55459
	0,5	0,17578	-0,08906	0,07822
	0	0,00000	0,00000	0,28916
	-0,5	-0,17578	0,08906	0,07822
	-1	-0,35156	-0,19688	-0,55459
-1	1	0,00000	-0,45000	0,00000
	0,5	0,00000	-0,60000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,60000	0,00000
	-1	0,00000	0,45000	0,00000

$\lambda=1$				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	3,00000	0,60000	0,00000
	0,5	1,50000	0,58125	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-1,50000	-0,58125	0,00000
	-1	-3,00000	-0,60000	0,00000
0,5	1	2,53125	-0,26250	-0,54844
	0,5	1,26563	0,00938	0,08438
	0	0,00000	0,00000	0,29531
	-0,5	-1,26563	-0,00938	0,08438
	-1	-2,53125	0,26250	-0,54844
0	1	1,50000	0,00000	-0,82500
	0,5	0,75000	0,00000	0,01875
	0	0,00000	0,00000	0,30000
	-0,5	-0,75000	0,00000	0,01875
	-1	-1,50000	0,00000	-0,82500
-0,5	1	0,46875	0,26250	-0,54844
	0,5	0,23438	-0,00938	0,08438
	0	0,00000	0,00000	0,29531
	-0,5	-0,23438	0,00938	0,08438
	-1	-0,46875	-0,26250	-0,54844
-1	1	0,00000	-0,60000	0,00000
	0,5	0,00000	-0,58125	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,58125	0,00000
	-1	0,00000	0,60000	0,00000

λ=5/4				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	3,75000	0,75000	0,00000
	0,5	1,87500	0,60000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-1,87500	-0,60000	0,00000
	-1	-3,75000	-0,75000	0,00000
0,5	1	3,16406	-0,32813	-0,54053
	0,5	1,58203	-0,05156	0,09229
	0	0,00000	0,00000	0,30322
	-0,5	-1,58203	0,05156	0,09229
	-1	-3,16406	0,32813	-0,54053
0	1	1,87500	0,00000	-0,86719
	0,5	0,93750	0,00000	-0,02344
	0	0,00000	0,00000	0,25781
	-0,5	-0,93750	0,00000	-0,02344
	-1	-1,87500	0,00000	-0,86719
-0,5	1	0,58594	0,32813	-0,54053
	0,5	0,29297	0,05156	0,09229
	0	0,00000	0,00000	0,30322
	-0,5	-0,29297	-0,05156	0,09229
	-1	-0,58594	-0,32813	-0,54053
-1	1	0,00000	-0,75000	0,00000
	0,5	0,00000	-0,60000	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,60000	0,00000
	-1	0,00000	0,75000	0,00000

λ=3/2				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	4,50000	0,90000	0,00000
	0,5	2,25000	0,63750	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-2,25000	-0,63750	0,00000
	-1	-4,50000	-0,90000	0,00000
0,5	1	3,79688	-0,39375	-0,53086
	0,5	1,89844	-0,10313	0,10195
	0	0,00000	0,00000	0,31289
	-0,5	-1,89844	0,10313	0,10195
	-1	-3,79688	0,39375	-0,53086
0	1	2,25000	0,00000	-0,91875
	0,5	1,12500	0,00000	-0,07500
	0	0,00000	0,00000	0,20625
	-0,5	-1,12500	0,00000	-0,07500
	-1	-2,25000	0,00000	-0,91875
-0,5	1	0,70313	0,39375	-0,53086
	0,5	0,35156	0,10313	0,10195
	0	0,00000	0,00000	0,31289
	-0,5	-0,35156	-0,10313	0,10195
	-1	-0,70313	-0,39375	-0,53086
-1	1	0,00000	-0,90000	0,00000
	0,5	0,00000	-0,63750	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,63750	0,00000
	-1	0,00000	0,90000	0,00000

λ=7/4				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	5,25000	1,05000	0,00000
	0,5	2,62500	0,68571	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-2,62500	-0,68571	0,00000
	-1	-5,25000	-1,05000	0,00000
0,5	1	4,42969	-0,45937	-0,51943
	0,5	2,21484	-0,14933	0,11338
	0	0,00000	0,00000	0,32432
	-0,5	-2,21484	0,14933	0,11338
	-1	-4,42969	0,45937	-0,51943
0	1	2,62500	0,00000	-0,97969
	0,5	1,31250	0,00000	-0,13594
	0	0,00000	0,00000	0,14531
	-0,5	-1,31250	0,00000	-0,13594
	-1	-2,62500	0,00000	-0,97969
-0,5	1	0,82031	0,45937	-0,51943
	0,5	0,41016	0,14933	0,11338
	0	0,00000	0,00000	0,32432
	-0,5	-0,41016	-0,14933	0,11338
	-1	-0,82031	-0,45937	-0,51943
-1	1	0,00000	-1,05000	0,00000
	0,5	0,00000	-0,68571	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,68571	0,00000
	-1	0,00000	1,05000	0,00000

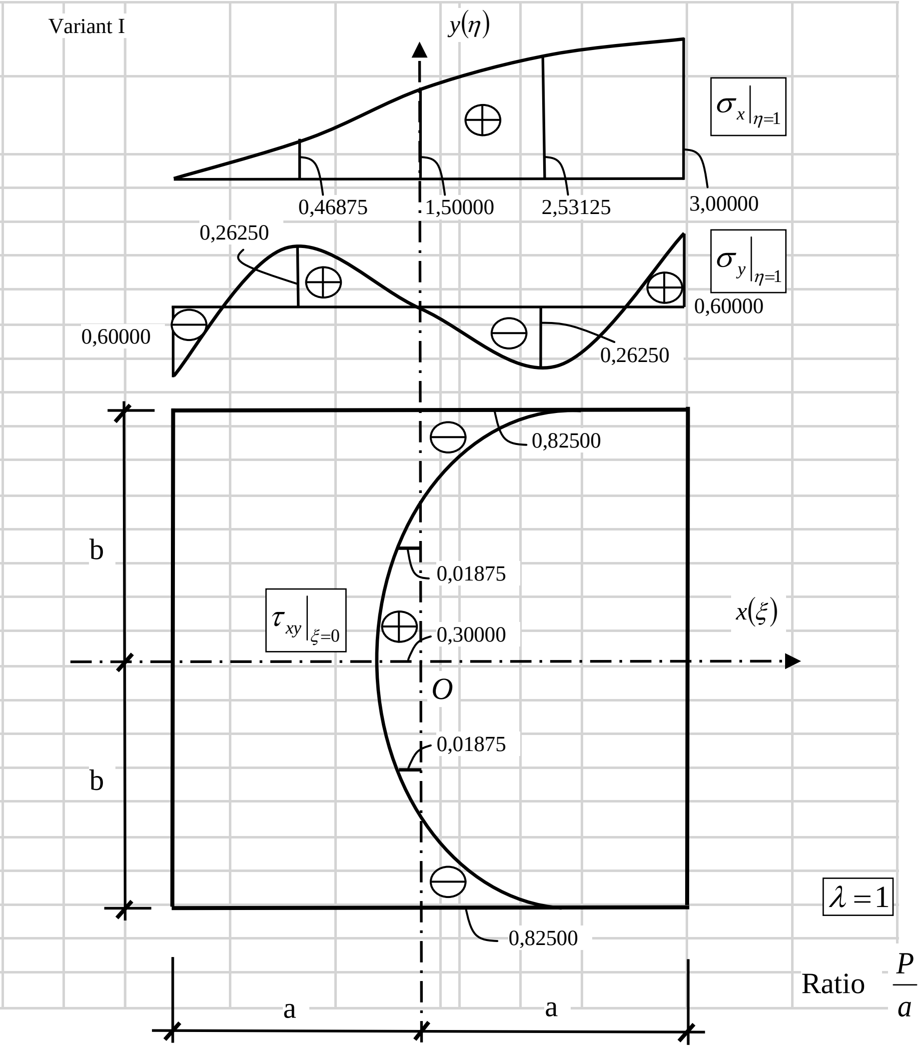
λ=2				
ξ	η	σ_x	σ_y	τ_{xy}
1	1	6,00000	1,20000	0,00000
	0,5	3,00000	0,74063	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-3,00000	-0,74063	0,00000
	-1	-6,00000	-1,20000	0,00000
0,5	1	5,06250	-0,52500	-0,50625
	0,5	2,53125	-0,19219	0,12656
	0	0,00000	0,00000	0,33750
	-0,5	-2,53125	0,19219	0,12656
	-1	-5,06250	0,52500	-0,50625
0	1	3,00000	0,00000	-1,05000
	0,5	1,50000	0,00000	-0,20625
	0	0,00000	0,00000	0,07500
	-0,5	-1,50000	0,00000	-0,20625
	-1	-3,00000	0,00000	-1,05000
-0,5	1	0,93750	0,52500	-0,50625
	0,5	0,46875	0,19219	0,12656
	0	0,00000	0,00000	0,33750
	-0,5	-0,46875	-0,19219	0,12656
	-1	-0,93750	-0,52500	-0,50625
-1	1	0,00000	-1,20000	0,00000
	0,5	0,00000	-0,74063	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,74063	0,00000
	-1	0,00000	1,20000	0,00000

λ=3

ξ	η	σ_x	σ_y	τ_{xy}
1	1	9,00000	1,80000	0,00000
	0,5	4,50000	0,99375	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-4,50000	-0,99375	0,00000
	-1	-9,00000	-1,80000	0,00000
0,5	1	7,59375	-0,78750	-0,43594
	0,5	0,00000	0,00000	0,40781
	0	0,00000	0,00000	0,40781
	-0,5	-3,79688	0,34688	0,19687
	-1	-7,59375	0,78750	-0,43594
0	1	4,50000	0,00000	-1,42500
	0,5	2,25000	0,00000	-0,58125
	0	0,00000	0,00000	-0,30000
	-0,5	-2,25000	0,00000	-0,58125
	-1	-4,50000	0,00000	-1,42500
-0,5	1	1,40625	0,78750	-0,43594
	0,5	0,70313	0,34688	0,19687
	0	0,00000	0,00000	0,40781
	-0,5	-0,70313	-0,34688	0,19687
	-1	-1,40625	-0,78750	-0,43594
-1	1	0,00000	-1,80000	0,00000
	0,5	0,00000	-0,99375	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	0,99375	0,00000
	-1	0,00000	1,80000	0,00000

λ=5

ξ	η	σ_x	σ_y	τ_{xy}
1	1	15,00000	3,00000	0,00000
	0,5	7,50000	1,55625	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	-7,50000	-1,55625	0,00000
	-1	-15,00000	-3,00000	0,00000
0,5	1	12,65625	-1,31250	-0,21094
	0,5	6,32813	-0,62812	0,42188
	0	0,00000	0,00000	0,63281
	-0,5	-6,32813	0,62812	0,42188
	-1	-12,65625	1,31250	-0,21094
0	1	7,50000	0,00000	-2,62500
	0,5	3,75000	0,00000	-1,78125
	0	0,00000	0,00000	-1,50000
	-0,5	-3,75000	0,00000	-1,78125
	-1	-7,50000	0,00000	-2,62500
-0,5	1	2,34375	1,31250	-0,21094
	0,5	1,17188	0,62812	0,42188
	0	0,00000	0,00000	0,63281
	-0,5	-1,17188	-0,62812	0,42188
	-1	-2,34375	-1,31250	-0,21094
-1	1	0,00000	-3,00000	0,00000
	0,5	0,00000	-1,55625	0,00000
	0	0,00000	0,00000	0,00000
	-0,5	0,00000	1,55625	0,00000
	-1	0,00000	3,00000	0,00000



Appendix II

Variant II | **Ratio P/a** | **Table 2**

λ=1/4

ξ	η	σ_x	σ_y	τ_{xy}
	1	0,09375	-0,12500	0,00000
	0,5	0,04688	0,01563	0,14063
1	0	0,00000	0,25000	0,18750
	-0,5	-0,04688	0,48438	0,14063
	-1	-0,09375	0,62500	0,00000
	1	0,05273	-0,04297	-0,01025
	0,5	0,02637	0,05664	0,09521
0,5	0	0,00000	0,25000	0,13037
	-0,5	-0,02637	0,44336	0,09521
	-1	-0,05273	0,54297	-0,01025
	1	0,02344	0,01563	-0,01172
	0,5	0,01172	0,08594	0,05859
0	0	0,00000	0,25000	0,08203
	-0,5	-0,01172	0,41406	0,05859
	-1	-0,02344	0,48438	-0,01172
	1	0,00586	0,05078	-0,00732
	0,5	0,00293	0,10352	0,02783
-0,5	0	0,00000	0,25000	0,03955
	-0,5	-0,00293	0,39648	0,02783
	-1	-0,00586	0,44922	-0,00732
	1	0,00000	0,06250	0,00000
	0,5	0,00000	0,10938	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	0,39063	0,00000
	-1	0,00000	0,43750	0,00000

λ=1/2

ξ	η	σ_x	σ_y	τ_{xy}
	1	0,37500	-0,50000	0,00000
	0,5	0,18750	-0,17188	0,28125
1	0	0,00000	0,25000	0,37500
	-0,5	-0,18750	0,67188	0,28125
	-1	-0,37500	1,00000	0,00000
	1	0,21094	-0,17188	-0,08203
	0,5	0,10547	-0,00781	0,12891
0,5	0	0,00000	0,25000	0,19922
	-0,5	-0,10547	0,50781	0,12891
	-1	-0,21094	0,67188	-0,08203
	1	0,09375	0,06250	-0,09375
	0,5	0,04688	0,10938	0,04688
0	0	0,00000	0,25000	0,09375
	-0,5	-0,04688	0,39063	0,04688
	-1	-0,09375	0,43750	-0,09375
	1	0,02344	0,20313	-0,05859
	0,5	0,01172	0,17969	0,01172
-0,5	0	0,00000	0,25000	0,03516
	-0,5	-0,01172	0,32031	0,01172
	-1	-0,02344	0,29688	-0,05859
	1	0,00000	0,25000	0,00000
	0,5	0,00000	0,20313	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	0,29688	0,00000
	-1	0,00000	0,25000	0,00000

λ=3/4

ξ	η	σ_x	σ_y	τ_{xy}
	1	0,84375	-1,12500	0,00000
	0,5	0,42188	-0,48438	0,42188
1	0	0,00000	0,25000	0,56250
	-0,5	-0,42188	0,98438	0,42188
	-1	-0,84375	1,62500	0,00000
	1	0,47461	-0,38672	-0,27686
	0,5	0,23730	-0,11523	0,03955
0,5	0	0,00000	0,25000	0,14502
	-0,5	-0,23730	0,61523	0,03955
	-1	-0,47461	0,88672	-0,27686
	1	0,21094	0,14063	-0,31641
	0,5	0,10547	0,14844	-0,10547
0	0	0,00000	0,25000	-0,03516
	-0,5	-0,10547	0,35156	-0,10547
	-1	-0,21094	0,35938	-0,31641
	1	0,05273	0,45703	-0,19775
	0,5	0,02637	0,30664	-0,09229
-0,5	0	0,00000	0,25000	-0,05713
	-0,5	-0,02637	0,19336	-0,09229
	-1	-0,05273	0,04297	-0,19775
	1	0,00000	0,56250	0,00000
	0,5	0,00000	0,35938	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	0,14063	0,00000
	-1	0,00000	-0,06250	0,00000

λ=1

ξ	η	σ_x	σ_y	τ_{xy}
	1	1,50000	-2,00000	0,00000
	0,5	0,75000	-0,92188	0,56250
1	0	0,00000	0,25000	0,75000
	-0,5	-0,75000	1,42188	0,56250
	-1	-1,50000	2,50000	0,00000
	1	0,84375	-0,68750	-0,65625
	0,5	0,42188	-0,26563	-0,23438
0,5	0	0,00000	0,25000	-0,09375
	-0,5	-0,42188	0,76563	-0,23438
	-1	-0,84375	1,18750	-0,65625
	1	0,37500	0,25000	-0,75000
	0,5	0,18750	0,20313	-0,46875
0	0	0,00000	0,25000	-0,37500
	-0,5	-0,18750	0,29688	-0,46875
	-1	-0,37500	0,25000	-0,75000
	1	0,09375	0,81250	-0,46875
	0,5	0,04688	0,48438	-0,32813
-0,5	0	0,00000	0,25000	-0,28125
	-0,5	-0,04688	0,01563	-0,32813
	-1	-0,09375	-0,31250	-0,46875
	1	0,00000	1,00000	0,00000
	0,5	0,00000	0,57813	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-0,07813	0,00000
	-1	0,00000	-0,50000	0,00000

λ=5/4				
ξ	η	σ_x	σ_y	τ_{xy}
	1	2,34375	-3,12500	0,00000
	0,5	1,17188	-1,48438	0,70313
1	0	0,00000	0,25000	0,93750
	-0,5	-1,17188	1,98438	0,70313
	-1	-2,34375	3,62500	0,00000
	1	1,31836	-1,07422	-1,28174
	0,5	0,65918	-0,45898	-0,75439
0,5	0	0,00000	0,25000	-0,57861
	-0,5	-0,65918	0,95898	-0,75439
	-1	-1,31836	1,57422	-1,28174
	1	0,58594	0,39063	-1,46484
	0,5	0,29297	0,27344	-1,11328
0	0	0,00000	0,25000	-0,99609
	-0,5	-0,29297	0,22656	-1,11328
	-1	-0,58594	0,10938	-1,46484
	1	0,14648	1,26953	-0,91553
	0,5	0,07324	0,71289	-0,73975
-0,5	0	0,00000	0,25000	-0,68115
	-0,5	-0,07324	-0,21289	-0,73975
	-1	-0,14648	-0,76953	-0,91553
	1	0,00000	1,56250	0,00000
	0,5	0,00000	0,85938	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-0,35938	0,00000
	-1	0,00000	-1,06250	0,00000

λ=3/2				
ξ	η	σ_x	σ_y	τ_{xy}
	1	3,37500	-4,50000	0,00000
	0,5	1,68750	-2,17188	0,84375
1	0	0,00000	0,25000	1,12500
	-0,5	-1,68750	2,67188	0,84375
	-1	-3,37500	5,00000	0,00000
	1	1,89844	-1,54688	-2,21484
	0,5	0,94922	-0,69531	-1,58203
0,5	0	0,00000	0,25000	-1,37109
	-0,5	-0,94922	1,19531	-1,58203
	-1	-1,89844	2,04688	-2,21484
	1	0,84375	0,56250	-2,53125
	0,5	0,42188	0,35938	-2,10938
0	0	0,00000	0,25000	-1,96875
	-0,5	-0,42188	0,14063	-2,10938
	-1	-0,84375	-0,06250	-2,53125
	1	0,21094	1,82813	-1,58203
	0,5	0,10547	0,99219	-1,37109
-0,5	0	0,00000	0,25000	-1,30078
	-0,5	-0,10547	-0,49219	-1,37109
	-1	-0,21094	-1,32813	-1,58203
	1	0,00000	2,25000	0,00000
	0,5	0,00000	1,20313	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-0,70313	0,00000
	-1	0,00000	-1,75000	0,00000

λ=7/4				
ξ	η	σ_x	σ_y	τ_{xy}
	1	4,59375	-6,12500	0,00000
	0,5	2,29688	-2,98438	0,98438
1	0	0,00000	0,25000	1,31250
	-0,5	-2,29688	3,48438	0,98438
	-1	-4,59375	6,62500	0,00000
	1	2,58398	-2,10547	-3,51709
	0,5	1,29199	-0,97461	-2,77881
0,5	0	0,00000	0,25000	-2,53271
	-0,5	-1,29199	1,47461	-2,77881
	-1	-2,58398	2,60547	-3,51709
	1	1,14844	0,76563	-4,01953
	0,5	0,57422	0,46094	-3,52734
0	0	0,00000	0,25000	-3,36328
	-0,5	-0,57422	0,03906	-3,52734
	-1	-1,14844	-0,26563	-4,01953
	1	0,28711	2,48828	-2,51221
	0,5	0,14355	1,32227	-2,26611
-0,5	0	0,00000	0,25000	-2,18408
	-0,5	-0,14355	-0,82227	-2,26611
	-1	-0,28711	-1,98828	-2,51221
	1	0,00000	3,06250	0,00000
	0,5	0,00000	1,60938	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-1,10938	0,00000
	-1	0,00000	-2,56250	0,00000

λ=2				
ξ	η	σ_x	σ_y	τ_{xy}
	1	6,00000	-8,00000	0,00000
	0,5	3,00000	-3,92188	1,12500
1	0	0,00000	0,25000	1,50000
	-0,5	-3,00000	4,42188	1,12500
	-1	-6,00000	8,50000	0,00000
	1	3,37500	-2,75000	-5,25000
	0,5	1,68750	-1,29688	-4,40625
0,5	0	0,00000	0,25000	-4,12500
	-0,5	-1,68750	1,79688	-4,40625
	-1	-3,37500	3,25000	-5,25000
	1	1,50000	1,00000	-6,00000
	0,5	0,75000	0,57813	-5,43750
0	0	0,00000	0,25000	-5,25000
	-0,5	-0,75000	-0,07813	-5,43750
	-1	-1,50000	-0,50000	-6,00000
	1	0,37500	3,25000	-3,75000
	0,5	0,18750	1,70313	-3,46875
-0,5	0	0,00000	0,25000	-3,37500
	-0,5	-0,18750	-1,20313	-3,46875
	-1	-0,37500	-2,75000	-3,75000
	1	0,00000	4,00000	0,00000
	0,5	0,00000	2,07813	0,00000
-1	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-1,57813	0,00000
	-1	0,00000	-3,50000	0,00000

λ=3

ξ	η	σ_x	σ_y	τ_{xy}
1	1	13,50000	-18,00000	0,00000
	0,5	6,75000	-8,92188	1,68750
	0	0,00000	0,25000	2,25000
	-0,5	-6,75000	9,42188	1,68750
	-1	-13,50000	18,50000	0,00000
0,5	1	7,59375	-6,18750	-17,71875
	0,5	3,79688	-3,01563	-16,45313
	0	0,00000	0,25000	-16,03125
	-0,5	-3,79688	3,51563	-16,45313
	-1	-7,59375	6,68750	-17,71875
0	1	3,37500	2,25000	-20,25000
	0,5	1,68750	1,20313	-19,40625
	0	0,00000	0,25000	-19,12500
	-0,5	-1,68750	-0,70313	-19,40625
	-1	-3,37500	-1,75000	-20,25000
-0,5	1	0,84375	7,31250	-12,65625
	0,5	0,42188	3,73438	-12,23438
	0	0,00000	0,25000	-12,09375
	-0,5	-0,42188	-3,23438	-12,23438
	-1	-0,84375	-6,81250	-12,65625
-1	1	0,00000	9,00000	0,00000
	0,5	0,00000	4,57813	0,00000
	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-4,07813	0,00000
	-1	0,00000	-8,50000	0,00000

λ=5

ξ	η	σ_x	σ_y	τ_{xy}
1	1	37,50000	-50,00000	0,00000
	0,5	18,75000	-24,92188	2,81250
	0	0,00000	0,25000	3,75000
	-0,5	-18,75000	25,42188	2,81250
	-1	-37,50000	50,50000	0,00000
0,5	1	21,09375	-17,18750	-82,03125
	0,5	10,54688	-8,51563	-79,92188
	0	0,00000	0,25000	-79,21875
	-0,5	-10,54688	9,01563	-79,92188
	-1	-21,09375	17,68750	-82,03125
0	1	9,37500	6,25000	-93,75000
	0,5	4,68750	3,20313	-92,34375
	0	0,00000	0,25000	-91,87500
	-0,5	-4,68750	-2,70313	-92,34375
	-1	-9,37500	-5,75000	-93,75000
-0,5	1	2,34375	20,31250	-58,59375
	0,5	1,17188	10,23438	-57,89063
	0	0,00000	0,25000	-57,65625
	-0,5	-1,17188	-9,73438	-57,89063
	-1	-2,34375	-19,81250	-58,59375
-1	1	0,00000	25,00000	0,00000
	0,5	0,00000	12,57813	0,00000
	0	0,00000	0,25000	0,00000
	-0,5	0,00000	-12,07813	0,00000
	-1	0,00000	-24,50000	0,00000